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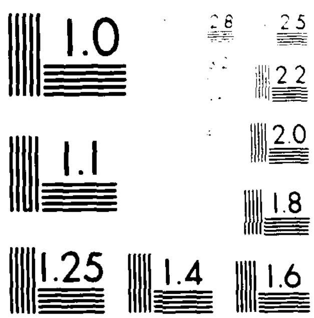
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SOLUTIONS OF THE POROUS MEDIUM EQUATION IN $\mathbb{R}^N$ UNDER OPTIMAL CONDITIONS ON INITIAL VALUES

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ABSTRACT

It is shown that the initial-value problem $u_t = \Delta(|u|^{m-1}u)$, $u(x,0) = u_0(x)$, where $m > 1$, has a solution on $\mathbf{R}^N \times [0,T)$ for some $T > 0$ if $R^{-(\frac{2}{m-1}+N)} \int_{\{|x|<R\}} |u_0(x)|dx$ is bounded independently of $R \geq 1$. The restriction on u_0 is known to be <u>necessary</u> as well by recent results of Aronson and Caffarelli, so this theorem is the best possible. Many supplementary results give refined estimates on the solution under various conditions on u_0, establish uniqueness within the existence class, allow u_0 to be a Radon measure, establish continuous dependence on u_0 in various spaces, etc.

AMS (MOS) Subject Classifications: 35K15, 35K65

Key Words: Porous media equation, initial-value problem, degenerate parabolic equation

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SIGNIFICANCE AND EXPLANATION

This work establishes existence of solutions of the initial-value problem $u_t = \Delta(|u|^{m-1}u)$, $u(x,0) = u_0(x)$, where $m > 1$, under the most general conditions on u_0. Namely, u_0 need only be such that

$R^{-(\frac{2}{m-1}+N)} \int_{\{|x|<R\}} |u_0(x)|dx$ is bounded independently of $R \geq 1$. Aronson and Caffarelli have shown this requirement to be necessary. Many auxiliary results are given in the form of estimates on solutions, uniqueness and continuous dependence theorems, etc.

While the results may be viewed as "technical" in that the main points consist of estimates of various sorts, the equation treated is of broad *practical interest* and the estimates reflect basic properties of the equation. The results obtained are the only ones known to the authors wherein the solvability of a realistic nonlinear initial value problem for a partial differential equation is now understood as completely as in the case of the heat equation.

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The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

SOLUTIONS OF THE POROUS MEDIUM EQUATION
IN $\mathbb{R}^N$ UNDER OPTIMAL CONDITIONS ON INITIAL VALUES

Philippe Bénilan, Michael G. Crandall and Michel Pierre

Introduction

This paper concerns the initial-value problem

$$\text{(IVP)} \qquad \begin{cases} u_t = \Delta(|u|^{m-1}u) \quad \text{on} \quad \mathbb{R}^N \times (0,T),\ m > 1 , \\ u(x,0) = u_0(x),\ x \in \mathbb{R}^N , \end{cases}$$

where the interval of existence $[0,T)$, $T > 0$, depends on the initial data u_0. We are interested in solving IVP for the largest possible class of functions u_0. In fact, we will prove a nonlinear version of the result which states that the linear problem

$$(0.1) \qquad \begin{cases} u_t = \Delta u \quad \text{on} \quad \mathbb{R}^N \times (0,T) , \\ u(x,0) = u_0(x),\ x \in \mathbb{R}^N \end{cases}$$

has a solution on some interval $[0,T)$ if

$$(0.2) \qquad \int |u_0(x)| e^{-c|x|^2} dx < \infty \quad \text{for some} \quad c > 0 .$$

(Of course, (0.2) guarantees that the solution formula for (0.1) provides a solution on $[0,4/c)$.) If $u_0 \geq 0$ it is know that (0.2) is also necessary for (0.1) to have a solution $u \geq 0$ on some time interval $[0,T)$, $T > 0$ ([15]).

Here we prove, without the aid of explicit solutions, that (IVP) has a solution on some interval $[0,T)$ if

$$(0.3) \qquad \sup_{R \geq 1} R^{-(N+\frac{2}{m-1})} \int_{\{|x|<R\}} |u_0(x)| dx < \infty ,$$

or, equivalently (see the Appendix),

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$$(0.3)' \qquad \sup_{R>1} \frac{1}{R^N} \int_{\{|x|<R\}} \frac{|u_0(x)|}{(1+|x|^2)^{1/(m-1)}}\, dx < \infty \ .$$

This condition is <u>necessary</u> in the class of nonnegative solutions as has been recently proved by Aronson and Caffarelli [16].

In fact, we will prove that if $\ell(u_0)$ is given by

$$(0.4) \qquad \ell(u_0) = \lim_{r\uparrow\infty} \sup_{R>r} R^{-(N+\frac{2}{m-1})} \int_{\{|x|<R\}} |u_0(x)|\,dx \ ,$$

then (IVP) has a solution on a maximal interval $[0,T(u_0))$ with

$$(0.5) \qquad T(u_0) \geq c/\ell(u_0)^{m-1}$$

where c is a constant depending only on N and m. This result implies (IVP) has a solution defined for all $t > 0$ if $\ell(u_0) = 0$ and (in the class of nonnegative solutions) we will show the necessity of $\ell(u_0) = 0$ for global-time existence.

Let us reconcile the above statements with several of the known solvability results for (IVP). If $u_0 \in L^1(\mathbb{R}^N)$, then $\ell(u_0) = 0$ and $T(u_0) = \infty$. The global time solvability in this case is known via the nonlinear semigroup theory (see [4], [6]). If $u_0 \in L^1(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$, again $\ell(u_0) = 0$ and $T(u_0) = \infty$. This is also obtainable from known results.

In the other direction, if

$$(0.6) \qquad u_0(x) = (\alpha + \beta|x|^2)^{\frac{1}{m-1}},\ \alpha,\beta > 0 \ ,$$

then the explicit solution

$$\begin{cases} u(x,t) = \left(\alpha \dfrac{\tau^k}{(\tau-t)^k} + \dfrac{k}{2mN} \dfrac{|x|^2}{\tau-t}\right)^{\frac{1}{m-1}} \ , \\ k = \dfrac{N(m-1)}{N(m-1)+2} \ , \quad \tau = \dfrac{k}{2mN} \dfrac{1}{\beta} \ , \end{cases}$$

shows u "blows up" at $k/2mN(\ell(u_0))^{m-1}$. This verifies that (0.5) is sharp as regards the functional dependence on $\ell(u_0)$. Further remarks concerning the sharpness of (0.5) and "blow up" may be found in the text.

The existence proof herein relies on rather novel estimates, the main one being an estimate of

$$(0.7)\qquad v(x,t) = \frac{u(x,t)}{(1+|x|^2)^{1/(m-1)}}$$

in $L^\infty(\mathbb{R}^N)$ for $t \in (0,T(u_0))$ in terms of the norm of u_0 defined by (0.3). In the process of obtaining this estimate we use the inequality established by Aronson and Benilan [1], namely

$$(0.8)\qquad \Delta u^{m-1} \geq -\frac{c}{t}\ , \quad c = c(m,N)$$

for certain nonnegative solutions of (IVP). We prove that if $u \geq 0$ satisfies (0.8) then $v = u/(1+|x|^2)^{1/(m-1)}$ can be estimated in $L^\infty(\mathbb{R}^N)$ by

$$(0.9)\qquad \sup_{R\geq r} R^{-(N+\frac{2}{m-1})} \int_{\{|x|\leq R\}} |u(x,t)|\,dx,\ r \geq 1\ ,$$

that is (see (0.3)') in terms of averages of v over balls centered at the origin. We then control the evolution of the quantity (0.9) with time.

The same type of proof also provides various L^∞-estimates of u in terms of various norms of u_0. We obtain, for instance, $u \in L^\infty(\mathbb{R}^N)$ for all $t > 0$ if

$$(0.10)\qquad \sup_{z\in\mathbb{R}^N} \int_{\{|x-z|\leq 1\}} |u_0(x)|\,dx < \infty\ .$$

The existence theorem is precisely formulated and proved in Section 1, as well as the L^∞-estimates mentioned above. The case in which u_0 is a measure satisfying the analog of (0.3) is also treated. Section 2 proves the uniqueness of the solutions obtained in Section 1. Some simple but useful remarks about the space X of functions u_0 satisfying (0.3) are collected in the Appendix.

Section 1. Existence

Throughout this section we will be concerned with the problem

$$\text{(IVP)}\qquad \begin{cases} u_t - \Delta(|u|^{m-1}u) = 0, \ t > 0, \ x \in \mathbb{R}^N , \\ u(x,0) = u_0(x) , \end{cases}$$

where

$$\text{(1.1)}\qquad N \geq 1 \quad \text{and} \quad m > 1 .$$

To formulate the main existence results for (IVP) we require several definitions. First, for each $f \in L^1_{\ell oc}(\mathbb{R}^N)$ and $r > 0$, let

$$\text{(1.2)}\qquad \|f\|_r = \sup_{R \geq r} R^{-(N+\frac{2}{m-1})} \int_{B_R} |f|$$

where $\int_K f$ denotes the Lebesque integral of f over $K \subset \mathbb{R}^N$ and $B_R = \{x \in \mathbb{R}^N; |x| < R\}$. If $K = \mathbb{R}^N$ we will simply write $\int f$. Note that if $\|f\|_r$ is finite for some $r > 0$, then it is finite for all $r > 0$. Set

$$\text{(1.3)}\qquad X = \{f \in L^1_{\ell oc}(\mathbb{R}^N); \|f\|_1 < \infty\}$$

and equip X with the norm $\| \ \|_1$. Clearly X is a Banach space and $\| \ \|_r$ is an equivalent norm on X for any $r > 0$. If $u_0 \in X$ we define

$$\text{(1.4)}\qquad \ell(u_0) = \lim_{r\uparrow\infty} \|u_0\|_r .$$

Next, for $\alpha \in \mathbb{R}$ define

$$\rho_\alpha(x) = (1 + |x|^2)^{-\alpha}$$

and let

$$L^1(\rho_\alpha) = \{f \in L^1_{\ell oc}(\mathbb{R}^N); \int |f|\rho_\alpha < \infty\}$$

be equipped with the norm

$$\|f\|_{L^1(\rho_\alpha)} = \int |f|\rho_\alpha .$$

We will be considering solutions of (IVP) as curves in X, $L^1(\rho_\alpha)$ and other spaces and will be writing "u(t)" in this context. The local-time existence theorem for (IVP) is stated next. It is made complex by the detailed information we have put in it. Let us present the theorem and then discuss its nature.

Theorem E

Existence: There is a constant $c_1 > 0$ depending only on N and m and a mapping

$$U : \{(t,u_0) : 0 \le t < T(u_0),\ u_0 \in X\} \to X$$

where

(1.5) $$T(u_0) = c_1/(\ell(u_0))^{m-1} \text{ and } T(u_0) = \infty \text{ if } \ell(u_0) = 0 ,$$

with the properties:

(i) If $u_0 \in X$, then $\|\rho_{1/(m-1)}U(t,u_0)\|_{L^\infty(\mathbb{R}^N)} \in L^\infty_{\ell oc}(0,T(u_0))$.

(ii) If $\alpha > \frac{1}{m-1} + \frac{N}{2}$ and $u_0 \in X$, then

$$U(\cdot,u_0) \in C([0,T(u_0));\ L^1(\rho_\alpha))$$

and $U(0,u_0) = u_0$.

(iii) $u(t) = U(t,u_0)$ is a solution of $u_t = \Delta(|u|^{m-1}u)$ in the sense of distributions on $\mathbb{R}^N \times (0,T(u_0))$.

Estimates: Let

(1.6) $$T_r(u_0) = c_1/\|u_0\|_r^{m-1} \quad \text{for } r \ge 1 .$$

These are constants c_2, c_3 depending only on N and m (and not on r) such that the following estimates hold:

(1.7) $$\begin{cases} \text{If } u_0 \in X,\ 1 \le r \le R \text{ and } t \in [0,T_r(u_0)) , \\ \text{then } \dfrac{\|U(t,u_0)\|_{L^\infty(B_R)}}{R^{2/(m-1)}} \le \dfrac{c_2}{t^\lambda}\|u_0\|_r^{2\lambda/N},\ \lambda = \dfrac{N}{(m-1)N+2} . \end{cases}$$

(1.8) $$\begin{cases} \text{If } u_0 \in X,\ r \ge 1, \text{ and } t \in [0,T_r(u_0)) , \\ \text{then } \|U(t,u_0)\|_r \le c_3\|u_0\|_r . \end{cases}$$

Dependence on Data: If $u_0, v_0 \in X$, $r \ge 1$, $\alpha \in \mathbb{R}$, $t \in [0,\min(T_r(u_0),T_r(v_0)))$, then

(1.9) $$\|U(t,u_0) - U(t,v_0)\|_{L^1(\rho_\alpha)} \le e^{B_1 t^{2\lambda/N}} \|u_0 - v_0\|_{L^1(\rho_\alpha)}$$

and

(1.10) $$\|U(t,u_0) - U(t,v_0)\|_r \le e^{B_2 t^{2\lambda/N}} \|u_0 - v_0\|_r$$

where B_1 depends only on $\max(\|u_0\|_r, \|v_0\|_r, \alpha, r)$, B_2 depends only on $\max(\|u_0\|_r, \|v_0\|_r)$, and λ is given in (1.7).

Ordering Principle: If $u_0, v_0 \in X$, then $u_0 \ge v_0$ implies $U(t,u_0) \ge U(t,v_0)$ for $0 \le t < \min(T(u_0), T(v_0))$.

Remark 1. If $u_0 \in X$, the assertions (i), (ii) of the theorem guarantee us a measurable representative $u(x,t)$, $x \in \mathbb{R}^N$, $t \in (0,T(u_0))$ of $U(\cdot,u_0)$ which is locally bounded in $\mathbb{R}^N \times (0,T(u_0))$. Thus u_t and $\Delta(|u|^{m-1}u)$ are defined in $\mathcal{D}'(\mathbb{R}^N \times (0,T(u_0)))$ and the assertion (iii) is meaningful.

In fact, one can prove regularity for $u = U(t,u_0)$ which we have not listed in this theorem. In the course of proving Proposition 1.6 below it is established that $(|u|^{m-1}u)_t$ and $\nabla(|u|^{m-1}u)$ are locally square integrable on $\mathbb{R}^N \times (0,T(u_0))$. Moreover, P. Sacks has proved (personal communication) that u is continuous on $\mathbb{R}^N \times (0,T(u_0))$.

Remark 2. The role played by $L^1(\rho_\alpha)$ is auxiliary: the natural claim would be "$U(\cdot,u_0) \in C([0,T(u_0)); X)$", but this is unfortunately false in general. One sees this from (i), since functions which are bounded by a multiple of $(1 + |x|^2)^{\frac{1}{m-1}}$ are not dense in X (see the Appendix on X). However, for $\alpha > \frac{1}{m-1} + \frac{N}{2}$ one has $X \subset L^1(\rho_\alpha)$ (see Lemma 1.1 below) and $U(\cdot,u_0)$ is continuous into $L^1(\rho_\alpha)$. The estimate (1.9) plays a significant role in the construction of U.

Remark 3. The estimates (1.7), (1.8) occur in the course of the proof and it is convenient to record them here. Note that (1.7) implies that $u(t) = U(t,u_0)$ satisfies

$$t^\lambda |u(x,t)| \le c_2 \|u_0\|_r^{\frac{2\lambda}{N}} \max(r^2, |x|^2)^{\frac{1}{m-1}}$$

so

(1.11) $$\sup_{x\in\mathbb{R}^N} \frac{|u(x,t)|}{(1+|x|^2)^{1/(m-1)}} \le \frac{c(r)}{t^\lambda} \|u_0\|_r^{\frac{2\lambda}{N}} \quad \text{for } 0 < t < T_r(u_0)$$

which implies (i) since $T(u_0) = \lim_{r\uparrow\infty} T_r(u_0)$. The same estimate also gives

$$(1.12)\qquad \frac{|u(x,t)|}{(1+|x|^2)^{1/(m-1)}} \le \frac{c_2}{t^\lambda} \|u_0\|_r^{\frac{2\lambda}{N}} \quad \text{for } 1 \le r \le |x|,\ 0 < t < T_r(u_0) \ .$$

We will use this consequence below.

<u>Remark 4</u>. The assertion (1.10) shows the continuity into X of the solution of (IVP) with respect to the initial data in X. It leads to the next result which exhibits a remarkable correlation between the global solvability of (IVP), (which is roughly equivalent to $\ell(u_0) = 0$ - see Remark 5) and the assumption of the initial value in the topology of X. Set

$$(1.13)\qquad X_0 = \{u_0 \in X;\ \ell(u_0) = 0\} \ .$$

<u>Corollary 1.1</u>. Let $u_0 \in X_0$. Then $U(t,u_0) \to u_0$ in X as $t \downarrow 0$. Moreover $U(t,u_0) \in X_0$ for $t > 0$ and $u(t) = U(t,u_0)$ satisfies

$$(1.14)\qquad \lim_{|x|\to\infty} \frac{u(x,t)}{(1+|x|^2)^{1/(m-1)}} = 0 \ .$$

<u>Remark 5</u>. This corollary is proved later, but we mention here that continuity into X for $u_0 \in X_0$ is essentially a consequence of (1.10) coupled with the fact (proved in the Appendix) that X_0 is the closure of $L^1(\mathbb{R}^N)$ in X and the known solvability of (IVP) for $u \in C([0,\infty):L^1(\mathbb{R}^N))$ when $u_0 \in L^1(\mathbb{R}^N)$. The result (1.14) follows from (1.12). Moreover, using the results of [2] one can prove that, in the nonnegative case, $\ell(u_0) = 0$ is <u>necessary</u> for global time solvability of (IVP). Indeed, it is proved in [2] that if $u(x,t)$ is a continuous nonnegative solution of $u_t = \Delta u^m$ on $[0,1)$, then

$$\int_{B_R} u_0(x)dx \le c[R^{(N+\frac{2}{m-1})} + u(0,1)^{1+(m-1)\frac{N}{2}}]$$

where $c = c(N,m)$. If $u(t)$ is defined on $(0,\infty)$, then $u_\beta(x,t) = u(\beta x, \beta^2 t)$ is also a solution for all $\beta > 1$. Hence

$$\frac{1}{\beta^N}\int_{B_{\beta R}} u_0(x)dx = \int_{B_R} u_0(\beta x)dx \le c[R^{(N+\frac{2}{m-1})} + u(0,\beta^2)^{1+(m-1)\frac{N}{2}}]$$

or, setting $S = \beta R$

$$\frac{1}{S^{(N+\frac{2}{m-1})}}\int_{B_S} u_0(x)dx \le c[\frac{1}{\beta^{\frac{2}{m-1}}} + \frac{\beta^N}{S^{(N+\frac{2}{m-1})}} u(0,\beta^2)^{1+(m-1)\frac{N}{2}}] \ .$$

We deduce that

$$\|u_0\|_r \le c\{\frac{1}{\beta^{\frac{1}{m-1}}} + c_\beta/r^{(N+\frac{2}{m-1})}\}$$

where $c_\beta = \beta^N u(0,\beta^2)^{(1+(m-1)\frac{N}{2})} < \infty$. Let $r \to \infty$ with β fixed and then $\beta \to \infty$ to conclude $\ell(u_0) = 0$.

Remark 6. In the scale of spaces $L^1(\rho_\alpha)$ we have $L^1(\rho_\alpha) \subset X_0$ when $\alpha < \frac{1}{m-1} + \frac{N}{2}$ and therefore $U(t,u_0)$ is defined for all $t > 0$ if $u_0 \in L^1(\rho_\alpha)$ for such α. (See the Appendix where other cases are also considered.) If $\gamma = \frac{1}{m-1} + \frac{N}{2}$, then $X \supset L^1(\rho_\gamma)$ but $U(t,u_0)$ will blow up in finite time for some $u_0 \in L^1(\rho_\gamma)$. A related statement is that no ball $\{u_0 \in L^1(\rho_\gamma) : \|u_0\|_{L^1(\rho_\gamma)} \le c\}$ is mapped into a bounded set in $L^\infty(B_1)$ by $U(t,\cdot)$ if t is large. This is clear from the family of explicit solutions (see Barenblatt [3] or Pattle [10])

$$u_n(x,t) = \frac{1}{(1+t)^\lambda}\{c_m(\frac{n^2}{4} - \frac{|x-x_n|^2}{(1+t)^{2\lambda/N}})\}_+^{\frac{1}{m-1}}$$

$$\lambda = \frac{N}{(m-1)N+2}, \quad c_m = \frac{\lambda(m-1)}{2mN}, \quad x_n = (n,0,\dots,0) .$$

(We implicitly assume the uniqueness result of Section 2 here.) Then

$$\int u_n(0) = \int_{\{\frac{n}{2} \le |x| \le \frac{3n}{2}\}} u_n(0) \le cn^{\frac{2}{m-1}+N}$$

so that

$$\|u_n(0)\|_{L^1(\rho_\gamma)} \le 4^\gamma c$$

while for $(1+t)^{2\lambda/N} > 8$

$$u_n(0,t) \ge \frac{1}{(1+t)^\lambda}[\frac{c_m}{8} n^2]^{\frac{1}{m-1}} .$$

Remark 7. Combining Theorem E with the uniqueness result of the next section, one can extend $U(t,u_0)$ uniquely to a maximal interval of existence $0 \le t < \tau(u_0)$ for each $u_0 \in X$ (see Theorem EU of Section 2). However, the dependence of $\tau(u_0)$ on $\ell(u_0)$

cannot have a better form than (1.5). This is shown by the example in the introduction. For a general nonnegative solution the computation in Remark 5 also shows that $\tau(u_0) \leq c/\ell(u_0)^{m-1}$.

<u>Proof of Theorem E.</u>

<u>Preliminaries</u>: As a launching point we will use that if $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, then (IVP) has a unique solution $u \in C([0,\infty); L^1(\mathbb{R}^N)) \cap L^\infty(\mathbb{R}^N \times [0,\infty))$ which satisfies the equation in the sense of distributions (see, e.g., [7], [6], [5]). Moreover, if

$$S : [0,\infty) \times (L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)) \to L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$$

is given by $S(t,u_0) = u(t)$ where $u(t)$ is the solution of (IVP) at time t, then for $u_0, v_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ we have:

(1.15) $$S(\cdot,u_0) \in C([0,\infty); L^1(\mathbb{R}^N))$$

(1.16) $$\|S(t,u_0) - S(t,v_0)\|_{L^1(\mathbb{R}^N)} \leq \|u_0 - v_0\|_{L^1(\mathbb{R}^N)} \quad \text{for } t \geq 0$$

(1.17) $$u_0 \leq v_0 \text{ implies } S(t,u_0) \leq S(t,v_0) \quad \text{for } t \geq 0$$

(1.18) $$\text{ess inf } u_0 \leq S(t,u_0) \leq \text{ess sup } u_0 \quad \text{for } t \geq 0$$

(1.19) $$\begin{cases} \text{If } u_0 \geq 0, \text{ then for } t > 0 \\ \Delta(S(t,u_0)^{m-1}) \geq -\frac{a}{t} \text{ in } \mathcal{D}'(\mathbb{R}^N) \\ \text{where } a = \frac{m-1}{m}\,\frac{N}{(m-1)N+2}. \end{cases}$$

The relations (1.15) - (1.18) are classical while (1.19) is a result of Aronson and Benilan [1].

We will also need the following technical result:

<u>Lemma 1.2.</u>

(i) For $\alpha > \frac{1}{m-1} + \frac{N}{2}$, we have $X \subset L^1(\rho_\alpha)$ with continuous injection.

(ii) If $r > 0$, $\{f_n\} \subset L^1_{\ell oc}(\mathbb{R}^N)$ and $f_n \to f$ in $L^1_{\ell oc}(\mathbb{R}^N)$, then

$$\|f\|_r \leq \liminf_{n\to\infty} \|f_n\|_r .$$

The lemma is proved in the Appendix.

Reduction to S: Here we observe that Theorem E will be proved if we can produce constants c_1, c_2, c_3, B_1, B_2 depending on the indicated quantities such that the assertions (1.7) - (1.10) hold when U is replaced by S and u_0, v_0 are chosen from $L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

Indeed, assume (1.7) - (1.10) hold for S. For $u_0 \in X$ define the truncations u_{0n} by

$$(1.22) \qquad \begin{cases} u_{0n}(x) = \tau_n(u_0(x)) & \text{if } |x| \le n , \\ u_{0n}(x) = 0 & \text{if } |x| > n , \end{cases}$$

where $\tau_n : \mathbb{R} \to \mathbb{R}$ is given by

$$\tau_n(s) = \begin{cases} s & \text{if } -n \le s \le n \\ n & \text{if } \quad n \le s \\ -n & \text{if } \quad s \le -n . \end{cases}$$

By (1.22) and Lemma 1.2 we have

$$(1.23) \qquad \begin{cases} u_{0n} \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \quad \text{for } n \ge 1 , \\ \|u_{0n}\|_r \text{ increases to } \|u_0\|_r \text{ and} \\ \|u_{0n} - u_0\|_{L^1(\rho_\alpha)} \to 0 \text{ as } n \to \infty \text{ if } \alpha > \frac{1}{m-1} + \frac{N}{2} . \end{cases}$$

Now $S(\cdot,u_{0n})$ is continuous from $[0,\infty)$ into $L^1(\mathbb{R}^N)$ and hence into $L^1(\rho_\alpha)$ for $\alpha > 0$. By (1.9), (assumed for S) and (1.23), $\{S(t,u_{0n})\}$ is a Cauchy sequence in $C([0,T_r(u_0)); L^1(\rho_\alpha))$ for all $r \ge 1$ and $\alpha > \frac{1}{m-1} + \frac{N}{2}$ and thus converges to a limit which we define to be $U(t,u_0)$. The function $U(\cdot,u_0)$ is then defined on $[0,T(u_0))$ (since $T_r(u_0)$ increases to $T(u_0)$) and clearly satisfies (ii), (iii), (1.7), (1.8), (1.9) and the ordering principle. By Remark 3, (1.7) implies (i). For (1.10), notice that $\|u_{0n} - v_{0n}\|_r \le \|u_0 - v_0\|_r$ and use Lemma 1.2. In this way we reduce the proof of Theorem E to the verification of (1.7) - (1.10) for S.

Proof of Estimates (1.7) and (1.8) for S. It will prove convenient to deal with a modification of $\|\cdot\|_r$. Let ϕ satisfy

$$(1.24)\quad \begin{cases} \phi(x) = \kappa(|x|) \text{ where } \kappa \in C_0^\infty([0,\infty)) \text{ is} \\ \text{nonincreasing, } \kappa(s) = 1 \text{ for } 0 \leq s \leq 1,\ \kappa(s) = 0 \text{ for } s \geq 2 \text{ and} \\ \kappa(s) = e^{-\frac{1}{2-s}} \text{ for } 2-\varepsilon < s < 2 \text{ and some } \varepsilon \in (0,1) \ . \end{cases}$$

The detailed structure of ϕ is for later convenience. Now define

$$(1.25)\qquad |f|_r = \sup_{R \geq r} \frac{1}{R^{\frac{2}{m-1}+N}} \int \phi(\frac{x}{R})|f| \quad \text{for} \quad r > 0 \quad .$$

One easily sees that $|\cdot|_r$ and $\|\cdot\|_r$ are equivalent norms on X with equivalence constants independent of $r \geq 1$.

The next result, which is related to our problem via (1.19), is the heart of our proofs.

<u>Proposition 1.3</u>. Let $u \in L^\infty(\mathbb{R}^N)$ be nonnegative, $\Lambda \in (0,\infty)$ and

$$(1.26)\qquad \Delta u^{m-1} \geq -\Lambda \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^N) \quad .$$

Then there is a constant K depending only on N and $m > 1$ such that for $1 \leq r \leq R$

$$(1.27)\qquad \frac{1}{R^2}\|u\|^{m-1}_{L^\infty(B_R)} \leq K(\Lambda^{\lambda(m-1)}|u|_r^{\frac{2\lambda(m-1)}{N}} + |u|_r^{m-1})$$

where $\lambda = N/((m-1)N+2)$.

In order to keep the general structure of the proof of Theorem E in view we postpone the proof of Proposition 1.3 until the end of this section. The next result yields the estimates (1.7), (1.8) of Theorem E for S (recall the equivalence of $|\ |_r$ and $\|\ \|_r$ uniformly in $r \geq 1$).

<u>Lemma 1.4</u>. There are constants c_4, c_5, $c_6 > 0$ depending only on N and $m > 1$ such that if $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $r \geq 1$ and $0 \leq t \leq c_4/|u_0|_r^{m-1}$, then

(i) $|S(t,u_0)|_r \leq c_5|u_0|_r$

and

$$\text{(ii)}\qquad \frac{\|S(t,u_0)\|^{m-1}_{L^\infty(B_R)}}{R^2} \leq \frac{c_6}{t^{\lambda(m-1)}}|u_0|_r^{\frac{2\lambda(m-1)}{N}} \quad \text{for} \quad R \geq r \quad .$$

Proof: We begin here the first of several computations which will have a "formal" appearance. That is, the computation is clearly valid only if the functions involved are sufficiently regular. We will not give detailed proofs here that the outcomes of these computations are valid for the less regular functions we deal with, as this may be done in a routine way during the construction of S by the reader's favorite method. In particular, one may use the representation of S by the generation theory of nonlinear semigroups and the results of [5] to do this (see, e.g., [6] for an example).

It will suffice to prove (i) and (ii) for $u_0 \geq 0$ since the order preserving property of S and $S(t,-u_0) = -S(t,u_0)$ imply

$$(S(t,u_0))^+ \leq S(t,u_0^+)$$

$$(S(t,u_0))^- \leq -S(t,-u_0^-) = S(t,u_0^-) \quad ,$$

where $r^+ = \max(r,0)$, $r^- = (-r)^+$. Thus $|S(t,u_0)| \leq S(t,u_0^+) + S(t,u_0^-)$. Thus we take $u_0 \geq 0$ below.

Formally $u = S(t,u_0)$ satisfies

$$\frac{\partial}{\partial t} \int u(x,t)\phi(\tfrac{x}{R}) = \int u_t\phi(\tfrac{x}{R}) = \int (\Delta u^m)\phi(\tfrac{x}{R})$$

$$= \int u^m \Delta\phi(\tfrac{x}{R}) = \int u^m \frac{1}{R^2} (\Delta\phi)(\tfrac{x}{R}) \quad .$$

We integrate this in time to conclude

$$\int u(x,t)\phi(\tfrac{x}{R}) = \int u_0(x)\phi(\tfrac{x}{R}) + \int_0^t \int \frac{u^m}{R^2} (\Delta\phi)(\tfrac{x}{R})$$

$$\leq \int u_0(x)\phi(\tfrac{x}{R}) + c \int_0^t R^{-2}\|u\|^{m-1}_{L^\infty(B_{2R})} \int_{B_{2R}} u$$

$$\leq \int u_0(x)\phi(\tfrac{x}{R}) + c \int_0^t R^{-2}\|u\|^{m-1}_{L^\infty(B_{2R})} \int u\phi(\tfrac{x}{2R}) \quad ,$$

where c denotes a constant varying from line to line. Multiply this inequality by $R^{-(\frac{2}{m-1}+N)}$ and take the supremum over $R \geq r$ of both sides to conclude that $g(t) = |u(t)|_r$ satisfies

$$g(t) \leq g(0) + c \int_0^t \left(\sup_{R \geq r} \frac{\|u\|^{m-1}_{L^\infty(B_R)}}{R^2}\right) g(\tau)\,d\tau \quad . \tag{1.28}$$

Using Proposition 1.3 in conjunction with (1.19) yields

$$\frac{\|u(t)\|^{m-1}_{L^\infty(B_R)}}{R^2} \leq c\left(\left(\frac{1}{t}\right)^{\lambda(m-1)} |u(t)|_r^{\frac{2\lambda(m-1)}{N}} + |u(t)|_r^{m-1}\right)$$

and using this in (1.28) we have

$$g(t) \leq g(0) + c \int_0^t \left(\frac{1}{\tau^{\lambda(m-1)}}\, g(\tau)^{\left(\frac{2\lambda(m-1)}{N}+1\right)} + g(\tau)^m\right) d\tau \quad . \tag{1.29}$$

A continuous solution of (1.29) lies below the solution h of

$$\begin{aligned} h'(t) &= c\left(\frac{1}{t^{\lambda(m-1)}}\, h(t)^{\left(1+\frac{2\lambda(m-1)}{N}\right)} + h(t)^m\right) \\ h(0) &= g(0) \quad , \end{aligned} \tag{1.30}$$

where this time c has the same meaning as in (1.29). To analyze (1.30) we consider the equation subject to

$$h(t)^m \leq \frac{1}{t^{\lambda(m-1)}}\, h(t)^{1+\frac{2\lambda(m-1)}{N}}$$

or

$$0 \leq t \leq 1/h(t)^{m-1}$$

which will be valid for t in some interval $[0,b]$. On this interval h is bounded above by the solution H of

$$H'(t) = \frac{2c}{t^{\lambda(m-1)}}\, H(t)^{1+\frac{2\lambda(m-1)}{N}} , \quad H(0) = g(0) \quad , \tag{1.31}$$

which is explicitly given by

$$H(t) = \left[g(0)^{-\frac{2\lambda(m-1)}{N}} - 2c(m-1)t^{\frac{2\lambda}{N}}\right]^{-\frac{N}{2\lambda(m-1)}} \quad . \tag{1.32}$$

We conclude that

$$g(t) \leq h(t) \leq H(t) \tag{1.33}$$

so long as

(1.34) $$0 \le \tau \le 1/H(\tau)^{m-1} \quad \text{for} \quad 0 \le \tau \le t \ .$$

By (1.32), (1.34) reduces to

$$0 \le \tau \le [g(0)^{-\frac{2\lambda(m-1)}{N}} - 2c(m-1)\tau^{\frac{2\lambda}{N}}]^{\frac{N}{2\lambda}}$$

or

$$0 \le (1+2c(m-1))\tau^{\frac{2\lambda}{N}} \le g(0)^{-(m-1)\frac{2\lambda}{N}}$$

or

(1.35) $$0 \le \tau \le c_4/g(0)^{m-1} \ .$$

Moreover, since H is increasing

(1.36) $$H(\tau) \le H(c_4/g(0)^{m-1}) = c_5\ g(0) \quad \text{for} \quad 0 \le \tau \le c_4/g(0)^{m-1}$$

for some constant c_5.

The validity of (1.33) and (1.36) on (1.35) yields (i). To obtain (ii), we use again Proposition 1.3 together with (1.19), (1.33), (1.36) to get

$$\begin{cases} \dfrac{1}{R^2} \|u\|^{m-1}_{L^\infty(B_R)} \le c[\dfrac{1}{t^{\lambda(m-1)}} |u_0|_r^{\frac{2\lambda(m-1)}{N}} + |u_0|_r^{m-1}] \\ \\ \text{for} \quad 0 \le t \le c_4/|u_0|_r^{m-1} \quad , \quad R \ge r \ge 1 \ . \end{cases}$$

But when $t|u_0|_r^{m-1} \le c_4$, we also have

$$|u_0|_r^{m-1} = |u_0|_r^{\frac{2\lambda(m-1)}{N}} |u_0|_r^{(1-\frac{2\lambda}{N})(m-1)} \le |u_0|_r^{\frac{2\lambda(m-1)}{N}} (\frac{c_4}{t})^{(1-\frac{2\lambda}{N})} \ ,$$

whence the estimate (ii) with a suitable c_6 (since $1 - \frac{2\lambda}{N} = \lambda(m-1)$).

<u>Proof of Estimates (1.9), (1.10) for S.</u>

To obtain (1.10) we formally proceed as follows: Let $u = S(t,u_0)$, $v = S(t,v_0)$. Let $p : \mathbb{R} \to \mathbb{R}$ be a smooth increasing function with $p(0) = 0$ and $j(r) = \int_0^r p(s)ds$. Then,

$$\begin{aligned} &\int \phi(\tfrac{x}{R})(\Delta(u^m - v^m))p(u^m - v^m) \\ &\quad = -\int \phi(\tfrac{x}{R})p'(u^m - v^m)|\nabla(u^m - v^m)|^2 - \int (\nabla\phi(\tfrac{x}{R}))p(u^m - v^m)\nabla(u^m - v^m) \\ &\quad \le -\int \nabla\phi(\tfrac{x}{R})\nabla j(u^m - v^m) = \int (\Delta\phi(\tfrac{x}{R}))j(u^m - v^m) \ . \end{aligned}$$

Now, let $p(r)$ tend to the signum function so that $j(u^m - v^m)$ tends to $|u^m - v^m|$. We conclude that

$$\int \phi(\tfrac{x}{R})\operatorname{sign}(u^m - v^m)\Delta(u^m - v^m) = \int \phi(\tfrac{x}{R})\operatorname{sign}(u-v)\Delta(u^m - v^m)$$
$$\leq \int \Delta\phi(\tfrac{x}{R})|u^m - v^m| \quad .$$

Using that u,v are solutions of $u_t = \Delta u^m$, we are led to

$$\frac{d}{dt}\int \phi(\tfrac{x}{R})|u-v| = \int \phi(\tfrac{x}{R})\operatorname{sign}(u-v)\Delta(u^m - v^m)$$
$$\leq \int |\Delta\phi(\tfrac{x}{R})|\ |u^m - v^m| \leq \int \frac{1}{R^2}|\Delta\phi|(\tfrac{x}{R})\max(m|u|^{m-1}, m|v|^{m-1})\ |u-v|$$

$$\leq c\max\left(R^{-2}\|u\|^{m-1}_{L^\infty(B_{2R})}, R^{-2}\|v\|^{m-1}_{L^\infty(B_{2R})}\right)\int_{B_{2R}} |u-v|$$

$$\leq c\max\left(R^{-2}\|u\|^{m-1}_{L^\infty(B_{2R})}, R^{-2}\|v\|^{m-1}_{L^\infty(B_{2R})}\right)\int \phi(\tfrac{x}{2R})\ |u-v| \quad .$$

Multiplication by $R^{-(\frac{2}{m-1}+N)}$, use of Lemma 1.4 (ii) and integration in time lead to the conclusion

$$|u(t) - v(t)|_r \leq |u_0 - v_0|_r + c(\max(|u_0|_r, |v_0|_r))^{\frac{2\lambda}{N}(m-1)}\int_0^t \frac{|u(\tau) - v(\tau)|_r}{\tau^{\lambda(m-1)}}\,d\tau$$

for $0 \leq t \leq c_4\min(|u_0|_r^{1-m}, |v_0|_r^{1-m})$. The result follows by comparison of $t \mapsto |u(t) - v(t)|_r$ with the solution of

$$\begin{cases} h'(t) = k\,h(t)t^{-\theta} \\ h(0) = |u_0 - v_0|_r \quad , \end{cases}$$

where $\theta = \lambda(m-1) < 1$ and $k = c(\max(|u_0|_r, |v_0|_r))^{\frac{2\lambda}{N}(m-1)}$, namely

$$h(t) = |u_0 - v_0|_r\, e^{\frac{k}{1-\theta}t^{1-\theta}} \quad .$$

We prove (1.8) in a similar way, using

$$\frac{d}{dt}\int \rho_\alpha|u-v| \leq \int (\Delta\rho_\alpha)|u^m - v^m|$$

which is obtained as above, while

$$\Delta\rho_\alpha = \Delta(1 + |x|^2)^{-\alpha} = \frac{-2\alpha}{(1 + |x|^2)^{\alpha+2}}(N + (N-2\alpha-2)|x|^2)$$

so that

$$|\Delta\rho_\alpha| \le C_\alpha \frac{\rho_\alpha}{1+|x|^2}$$

and

$$\frac{d}{dt}\int \rho_\alpha |u-v| \le C_\alpha m \int \frac{\max(|u|^{m-1}, |v|^{m-1})}{1+|x|^2} |u-v|\rho_\alpha \ .$$

But, for $r \le R \le |x| \le 2R$

$$\frac{|u(x,t)|^{m-1}}{1+|x|^2} \le \frac{(2R)^2}{1+R^2} \frac{\|u(t)\|^{m-1}_{L^\infty(B_{2R})}}{(2R)^2}$$

and for $|x| \le r$

$$\frac{|u(x,t)|^{m-1}}{1+|x|^2} \le r^2 \frac{\|u(t)\|^{m-1}_{L^\infty(B_r)}}{r^2}$$

so that

$$\sup_{x\in\mathbb{R}^N} \frac{|u(x,t)|^{m-1}}{1+|x|^2} \le c_r \sup_{R\ge r} \frac{\|u(t)\|^{m-1}_{L^\infty(B_R)}}{R^2} \ .$$

Then we can proceed as above.

The proof of Theorem E is now completed by proving Proposition 1.3.

Proof of Proposition 1.3. We will give here our original proof of this result using Moser's well-known ideas. It is convenient for our purposes and entirely self-contained. See, however, the remarks at the end of the proof.

Let u satisfy the assumptions of the proposition. Let us also assume that u is smooth and strictly positive (so that, in particular, u^{m-1} is smooth). We will indicate later how to get rid of this extra assumption.

Let $\psi \in C_0^\infty(\mathbb{R}^N)$, $\psi \ge 0$. Then

$$\Delta(\psi u)^{m-1} = \psi^{m-1}\Delta u^{m-1} + 2\,\nabla\psi^{m-1}\nabla u^{m-1} + u^{m-1}\Delta\psi^{m-1}$$

and by (1.26)

$$\Delta(\psi u)^{m-1} \ge -\Lambda\psi^{m-1} + 2\,\nabla\psi^{m-1}\nabla u^{m-1} + u^{m-1}\Delta\psi^{m-1} \ . \tag{1.37}$$

We will only use $\psi(x) = \phi(x/R)$ for $R > 0$ with ϕ as in (1.24). In this case $\psi^\theta \in C_0^\infty(\mathbb{R}^N)$ for every $\theta > 0$, so the regularity of ψ^{m-1} is not in question. Moreover, one easily checks that

$$\lim_{|x|\uparrow 2} \phi(x)^{p-2}|\Delta\phi(x)| + \phi(x)^{p-3}|\nabla\phi(x)|^2 = 0$$

for all $p > 1$ and hence

$$(1.38) \qquad \|\phi^{p-2}|\Delta\phi| + \phi^{p-3}|\nabla\phi|^2\|_{L^\infty(\mathbb{R}^N)} < c_p < +\infty \ .$$

Now, multiply (1.37) by $(\psi u)^p$ where $p > 1$ and integrate to find

$$(1.39) \qquad \begin{cases} \int \nabla(\psi u)^p \nabla(\psi u)^{m-1} < \Lambda \int \psi^{m-1+p} u^p \\ \quad - 2\int (\psi u)^p \nabla\psi^{m-1}\nabla u^{m-1} - \int (\psi u)^p u^{m-1}\Delta\psi^{m-1} \ . \end{cases}$$

We rewrite the various terms. One has

$$(1.40) \qquad \int \nabla(\psi u)^p \nabla(\psi u)^{m-1} = \frac{4(m-1)p}{(m-1+p)^2}\int \left|\nabla(\psi u)^{\frac{p+m-1}{2}}\right|^2$$

$$(1.41) \qquad \int (\psi u)^p \nabla\psi^{m-1}\nabla u^{m-1} = \frac{(m-1)^2}{(m-1+p)^2}\int \nabla\psi^{m-1+p}\nabla u^{m-1+p}$$

$$= -\frac{(m-1)^2}{(m-1+p)^2}\int u^{m-1+p}\,\Delta\psi^{m-1+p}$$

$$= -\frac{(m-1)^2}{(m-1+p)}\int u^{m-1}(u\psi)^p[(m-2+p)\psi^{m-3}|\nabla\psi|^2 + \psi^{m-2}\Delta\psi] \ .$$

Now we put $\psi(x) = \phi(x/R)$ in (1.39), (1.40), (1.41), we use (1.38) and the fact that $\Delta\phi^{m-1}$ is bounded to obtain

$$(1.42) \qquad \int \left|\nabla(\phi(\tfrac{x}{R})u)^{\frac{p+m-1}{2}}\right|^2 < cp[\Lambda\int (\phi(\tfrac{x}{R})u)^p + \frac{1}{R^2}\int (\phi(\tfrac{x}{R})u)^p u^{m-1}]$$

where c is a constant independent of p and R.

Fix $r < 1$ and let

$$(1.43) \qquad A = \sup_{R > r} \frac{\|u\|^{m-1}_{L^\infty(B_R)}}{R^2} \ .$$

Then, (1.42), (1.43) yield, with a new constant,

$$(1.44) \qquad \int \left|\nabla(\phi(\tfrac{x}{R})u)^{\frac{p+m-1}{2}}\right|^2 < cp(\Lambda + A)\int (\phi(\tfrac{x}{R})u)^p \quad \text{for } R > r \ .$$

Due to the use of Sobolev's inequalities, our proof will now depend on the dimension N. Let us assume first that $N > 3$. Then

(1.45) $$\int \left|\nabla(\phi(\tfrac{x}{R})u)^{\frac{p+m-1}{2}}\right|^2 \geq c\left[\int (\phi(\tfrac{x}{R})u)^{\frac{2^*}{2}(p+m-1)}\right]^{\frac{2}{2^*}}$$

where $2^* = 2N/(N-2)$. Combining (1.45) with (1.44) leads to

(1.46) $$\left[\int [\phi(\tfrac{x}{R})u]^{sp+b}\right]^{1/s} \leq cp(\Lambda+A) \int [\phi(\tfrac{x}{R})u]^p \quad ,$$

where

(1.47) $$s = 2^*/2 = N/(N-2) \quad , \; b = s(m-1) = (m-1)N/(N-2) \quad .$$

Multiplying (1.46) by $R^{-\gamma(p+m-1)}$ (where $\gamma > 0$ will be discussed shortly), and taking s^{th} powers one finds

(1.48) $$\frac{1}{R^{\gamma(sp+b)}} \int [\phi(\tfrac{x}{R})u]^{sp+b} \leq c^s p^s (\Lambda+A)^s \left[\frac{1}{R^{\gamma(p+m-1)}} \int [\phi(\tfrac{x}{R})u]^p\right]^s \quad .$$

Define

(1.49) $$\begin{cases} p_0 = 1, \; p_{k+1} = sp_k + b \quad \text{for} \quad k = 0,1,\ldots, \\[2ex] \theta_0 = \dfrac{1}{m-1} + N \quad , \quad \theta_{k+1} = \dfrac{p_k}{p_k+m-1}\,\theta_k = s\,\dfrac{p_k}{p_{k+1}}\,\theta_k \quad \text{for} \quad k = 0,1,\ldots, \\[2ex] a_k = \sup\limits_{R \geq r} \dfrac{1}{R^{\theta_k p_k}} \displaystyle\int (\phi(\tfrac{x}{R})u)^{p_k} \quad \text{for} \quad k = 0,1,\ldots, \quad . \end{cases}$$

Now put $p = p_k$, $\gamma = \theta_{k+1}$ in (1.48) and take the supremum over $R \geq r$ in the result to obtain

(1.50) $$a_{k+1} \leq c^s p_k^s (\Lambda+A)^s a_k^s \quad .$$

Iterating (1.49) yields

$$a_{k+1} \leq [c(\Lambda+A)]^{s+s^2+\cdots+s^{k+1}} \, p_k^s p_{k-1}^{s^2} \cdots p_0^{s^{k+1}} \, (a_0)^{s^{k+1}} \quad ,$$

or

(1.51) $$(a_{k+1})^{1/p_{k+1}} \leq [c(\Lambda+A)]^{\alpha_k} \, M_k (a_0)^{\beta_k}$$

where

(1.52) $$\begin{cases} \alpha_k = \dfrac{s+s^2+\cdots+s^{k+1}}{p_{k+1}} \\[2ex] M_k = \left(p_k^s p_{k-1}^{s^2} \cdots p_0^{s^{k+1}}\right)^{1/p_{k+1}} \\[2ex] \beta_k = s^{k+1}/p_{k+1} \end{cases}$$

and by (1.47), (1.49)

$$p_{k+1} = sp_k + b = s^{k+1} + (m-1)(s^{k+1} + s^k + \cdots + s) \quad . \tag{1.53}$$

From (1.52), (1.53), (1.49) one easily deduces

$$\begin{cases} \lim\limits_{k\to\infty} \alpha_k = \dfrac{s}{ms-1} = \dfrac{N}{N(m-1)+2} = \lambda \quad , \\ \\ \lim\limits_{k\to\infty} \beta_k = \dfrac{ms-1}{s-1} = \dfrac{2}{N(m-1)+2} = \dfrac{2\lambda}{N} \quad , \\ \\ \limsup\limits_{k\to\infty} M_k < \infty \quad , \\ \\ \lim\limits_{k\to\infty} \theta_k = \lim\limits_{k\to\infty} \dfrac{s^k}{p_k}\left(\dfrac{2}{m-1} + N\right) = \dfrac{2}{m-1} \quad . \end{cases} \tag{1.54}$$

Using this information in (1.51) and recalling (1.49), we conclude that, with a new c,

$$\frac{1}{R^{\frac{2}{m-1}}} \|\phi(\tfrac{x}{R})u\|_{L^\infty(B_R)} \le c(\Lambda+A)^\lambda |u|_r^{2\lambda/N}$$

for $R \ge r$. Remembering (1.43) this gives

$$A^{\frac{1}{m-1}} \le c(\Lambda+A)^\lambda \ |u|_r^{2\lambda/N} \quad . \tag{1.55}$$

Finally, we analyze this implicit inequality.

<u>Lemma 1.3</u>. Let $0 < \beta < \alpha$ and $A, \Lambda, L > 0$ satisfy

$$A^\alpha \le L(\Lambda + A)^\beta \quad . \tag{1.56}$$

Then there is a constant K depending only on α, β such that

$$A \le K(\Lambda^{\beta/\alpha} L^{1/\alpha} + L^{1/(\alpha-\beta)}) \quad . \tag{1.57}$$

<u>Proof</u>. Assume $A \ge \Lambda$. Then

$$A^\alpha \le L(\Lambda + A)^\beta \le 2^\beta L A^\beta$$

and

$$A \le (2^\beta L)^{1/(\alpha-\beta)} \quad .$$

If $A \le \Lambda$, then $A^\alpha \le 2^\beta L \Lambda^\beta$ and

$$A \le 2^{\beta/\alpha} L^{1/\alpha} \Lambda^{\beta/\alpha} \quad ,$$

whence the result.

Using the lemma, (1.55) implies, with a new C,

$$A \le C[\Lambda^{\lambda(m-1)} |u|_r^{\frac{2\lambda(m-1)}{N}} + |u|_r^{m-1}] ,$$

and we have (1.27) when $N \ge 3$.

For $\underline{N = 2}$, we start from (1.44) and proceed as follows: By the inequality

$$\|v\|_{L^{1^*}(\mathbb{R}^N)} \le c\|\nabla v\|_{L^1(\mathbb{R}^N)} , \quad 1^* = \frac{N}{N-1} = 2 \quad \text{for} \quad N = 2 .$$

applied to v^s, $s \ge 1$ and Hölder's inequality we have

$$\|v^s\|_{L^{1^*}(\mathbb{R}^N)} \le c \int |\nabla v^s| = cs \int |v^{s-1}\nabla v| \le cs(\int v^{2(s-1)})^{1/2}(\int |\nabla v|^2)^{1/2} ,$$

or

$$\int |\nabla v|^2 \ge \frac{c}{s^2} \frac{\int v^{2s}}{\int v^{2(s-1)}} .$$

Apply this with $v = (\phi(\frac{x}{R})u)^{\frac{p+m-1}{2}}$ in (1.44) to obtain

$$\int [\phi(\tfrac{x}{R})u]^{s(p+m-1)} \le cs^2p(\Lambda + A)[\int (\phi(\tfrac{x}{R})u)^p][\int (\phi(\tfrac{x}{R})u)^{(s-1)(p+m-1)}] .$$

Choose $s = 1 + \frac{p}{p+m-1}$, so that $s(p+m-1) = 2p+m-1$ and the above inequality becomes

$$\int (\phi(\tfrac{x}{R})u)^{2p+m-1} \le cs^2p(\Lambda + A) [\int (\phi(\tfrac{x}{R})u)^p]^2$$

or, with a new constant,

$$(\int (\phi(\tfrac{x}{R})u)^{2p+m-1})^{1/2} \le cp^{1/2}\sqrt{\Lambda + A} \int (\phi(\tfrac{x}{R})u)^p . \tag{1.58}$$

This is treated analogously to (1.46). If we set

$$\begin{cases} p_0 = 1, \ p_{k+1} = 2p_k + m - 1 \quad \forall k \ge 0 \\ \theta_0 = \frac{2}{m-1} + 2, \ \theta_{k+1} = 2\frac{p_k}{p_{k+1}}\theta_k \\ a_k = \sup\limits_{R \ge r} \frac{1}{R^{\theta_k p_k}} \int (\phi(\frac{x}{R})u)^{p_k} \end{cases} \tag{1.59}$$

then, by (1.58),

$$a_{k+1} \le c^2p_k(\Lambda + A)a_k^2 .$$

This relation is similar to (1.50) where s is replaced by 2 and we obtain in the same way

$$a_{k+1}^{\frac{1}{p_{k+1}}} \leq c(\sqrt{\Lambda + A})^{\alpha_k}(a_0)^{\beta_k} \tag{1.60}$$

where (see (1.52) - (1.54))

$$p_{k+1} = 2^{k+1} + (m-1)(2^k + 2^{k-1} + \ldots + 1)$$

$$\lim_{k\to\infty} \alpha_k = \lim_{k\to\infty} \frac{2+2^2+\ldots+2^{k+1}}{p_{k+1}} = \frac{2}{m}$$

$$\lim_{k\to\infty} \beta_k = \lim_{k\to\infty} 2^{k+1}/p_{k+1} = \frac{1}{m}$$

$$\lim_{k\to\infty} \theta_k = \lim_{k\to\infty} 2^k\theta_0/p_k = \frac{1}{m}\left(\frac{2}{m-1} + 2\right) = \frac{2}{m-1} \ .$$

After letting k tend to ∞ in (1.60), we have

$$\Lambda^{\frac{1}{m-1}} \leq c(\Lambda + A)^{1/m} |u|_r^{\frac{1}{m}}$$

which is exactly (1.55) with $N = 2$ (recall that then $\lambda = \frac{2}{(m-1)2+2} = \frac{1}{m}$). The proof is completed as in the case $N \geq 3$.

For $\underline{N = 1}$, we use only one step starting from the embedding

$$\|v\|_{L^\infty(\mathbb{R})} \leq c\|\nabla v\|_{L^1(\mathbb{R})} \ .$$

For all $s \geq 1$, we have

$$\|v^s\|_{L^\infty(\mathbb{R})} \leq c \int |\nabla v^s| \leq cs \left(\int v^{2(s-1)}\right)^{1/2}\left(\int |\nabla v|^2\right)^{1/2} \ .$$

We use this in (1.44) with $v = (\phi(\frac{x}{R})u)^{\frac{p+m-1}{2}}$ and $p = 1$ to obtain

$$\|\phi(\tfrac{x}{R})u\|_{L^\infty(\mathbb{R})}^{ms} \leq cs^2(\Lambda + A)\left(\int [\phi(\tfrac{x}{R})u]^{m(s-1)}\right)\left(\int \phi(\tfrac{x}{R})u\right) \ .$$

Now choose $s = 1 + \frac{1}{m}$ and this becomes

$$\|\phi(\tfrac{x}{R})u\|_{L^\infty(\mathbb{R})}^{m+1} \leq c(\Lambda + A)\left(\int \phi(\tfrac{x}{R})u\right)^2 \ . \tag{1.61}$$

We multiply this inequality by $R^{-2(\frac{2}{m-1}+1)}$, take the supremum over $R \geq r$ and obtain

$$\left(\frac{\|u\|^{m-1}_{L^\infty(B_R)}}{R^2}\right)^{\frac{m+1}{m-1}} \leq c(\Lambda + A)\ |u|_r^2$$

or

$$A^{\frac{1}{m-1}} \leq c(\Lambda + A)^{\frac{1}{m+1}}\ |u|_r^{\frac{2}{m+1}}\ ,$$

which is exactly (1.55) when $N = 1$ (then $\lambda = \frac{1}{m+1}$) .

To complete the proof of Proposition 1.1, we now indicate how to drop the assumption that u is smooth and positive. If u satisfies the assumption of Proposition 1.1, we introduce $v_\varepsilon > 0$ defined by $v_\varepsilon^{m-1} = \rho_\varepsilon * u^{m-1} + \varepsilon$ where ρ_ε is a standard mollifier and $*$ denotes the convolution in $\mathbb{R}^N$. Then v_ε is bounded, smooth, positive and

$$\Delta v_\varepsilon^{m-1} = \rho_\varepsilon * \Delta u^{m-1} \geq -\Lambda\ .$$

Hence above computations are valid with v_ε in place of u. Since v_ε converges a.e. to u when ε tends to 0 and is uniformly locally bounded, passing to the limit in (1.46) - written with v_ε in place of u - yields (1.46) for u itself (one easily checks that $A(v_\varepsilon)$ converges to $A(u)$ when $u \in L^\infty(\mathbb{R}^N)$). The computations coming after (1.46) do not require any smoothness of u so that one can complete the proof exactly as above for $N \geq 3$. If $N = 2,1$, we may pass to the limit in (1.58) and (1.61) as $a \downarrow 0$, etc.

<u>Remarks on Proposition 1.3</u>. H. Brezis and others suggested that this result might correspond to a simple interior estimate. In fact, this is the case. For example, if $Lu = -(a_{ij}(x)u_{x_i})_{x_j}$ is a divergence form uniformly elliptic operator with coefficients in $L^\infty(B_{2R})$ and $u \geq 0$ satisfies $Lu \leq f$ in B_{2R} where $f \in L^p(B_{2R})$ with $p > N/2$, then

$$\|u\|_{L^\infty(B_R)} \leq C\left(\|f\|^\alpha_{L^p(B_{2R})}\ \|u\|^{1-\alpha}_{L^{p_0}(B_{2R})} + R^{-N/p_0}\ \|u\|_{L^{p_0}(B_{2R})}\right)$$

where C depends suitably on the coefficients of L, p and $p_0 > 0$ and $\alpha = Np/(p_0(2p-N) + Np)$.

Proposition 1.3 follows easily. The proof of this result can be given using standard results and methods of Moser, etc. This will be done in generality (i.e., in the parabolic case with lower order terms, etc.) elsewhere.

Proof of Corollary 1.1

Let $u_0 \in X_0$ and define u_{0n} as in (1.22). Then, for all $r > 0$,

$$\lim_{n\to\infty} \|u_{0n} - u_0\|_r = 0 \quad . \tag{1.62}$$

Indeed, if $\gamma = \frac{2}{m-1} + N$, then for all $r \le R$ and $r \le r_0$ we have

$$\frac{1}{R^\gamma} \int_{\{|x|\le R\}} |u_0 - u_{0n}| \le \frac{1}{R^\gamma} [\int_{\{|x|\le r_0\}} |u_0 - u_{0n}| + 2 \int_{\{r_0 \le |x| \le R\}} |u_0|] \quad .$$

Taking the supremum over $R \ge r$ with r, r_0 fixed yields

$$\|u_{0n} - u_0\|_r \le \frac{1}{r^\gamma} \int_{\{|x|\le r_0\}} |u_0 - u_{0n}| + 2\|u_0\|_{r_0}$$

so that (see (1.22))

$$\limsup_{n\to\infty} \|u_{0n} - u_0\|_r \le 2\|u_0\|_{r_0} \quad .$$

Since $\lim_{r_0\to\infty} \|u_0\|_{r_0} = 0$, we obtain (1.62).

Now, $S(\cdot,u_{0n}) \in C([0,\infty); L^1(\mathbb{R}^N)) \hookrightarrow C([0,\infty); X)$. By (1.62) and the estimate (1.10) of Theorem E, $S(\cdot,u_{0n})$ is a Cauchy sequence in $C([0,T_r(u_0)); X)$ for all $r > 0$. Since $\lim_{r\to\infty} T_r(u_0) = +\infty$, $U(\cdot,u_0)$, obtained as the limit of $S(\cdot,u_{0n})$, is defined on $[0,\infty)$ and belongs to $C([0,\infty); X)$.

Since X_0 is closed in X (see Appendix), $u(t) \in X_0$ for all $t \ge 0$. The more precise estimate (1.14) comes from (1.6), (1.12).

Theorem E asserts the existence of a solution $u(t)$ of (IVP) for $u_0 \in X$ and claims at the same time that $\frac{u(t)}{(1+|x|^2)^{1/(m-1)}} \in L^\infty(\mathbb{R}^N)$ for $t \in (0,T(u_0))$. The next result shows that this L^∞-estimate can be improved when u_0 is further restricted. The proof requires only small changes in the arguments above.

For $\mu \in (0,1]$, we set

$$(1.63)\qquad \|f\|_{r,\mu} = \sup_{R\geq r} \frac{1}{R^{\mu(\frac{2}{m-1}+N)}} \int_{B_R} |f| \quad ,$$

so that $\|\cdot\|_r = \|\cdot\|_{r,1}$. Set

$$X^\mu = \{f \in L^1_{\ell oc}(\mathbb{R}^N);\ \|f\|_{r,\mu} < \infty \ \text{ for some } \ r > 0\} \quad .$$

Obviously $X^\mu \subset X$ *for all* $\mu \in (0,1]$.

Proposition 1.5. Let $\mu \in (0,1)$ and $u_0 \in X^\mu$. Then

$$(1.65)\qquad U(t,u_0)\rho_{\mu/(m-1)} \in L^\infty(\mathbb{R}^N) \quad \text{for} \quad 0 < t < +\infty \quad .$$

Moreover, if

$$(1.66)\qquad T_{r,\mu}(u_0) = c_1/\|u_0\|^{m-1}_{r,\mu}$$

then

$$(1.67)\qquad \begin{cases} \text{for} \quad R \geq r \geq 1 \quad \text{and} \quad 0 < t < T_{r,\mu}(u_0) \\ \dfrac{\|U(t,u_0)\|_{L^\infty(B_R)}}{R^{2\mu/(m-1)}} \leq \dfrac{c_2}{t^\lambda} \|u_0\|^{2\lambda/N}_{r,\mu},\ \lambda = \dfrac{N}{(m-1)N+2} \end{cases} ,$$

and the estimates (1.8), (1.10) in Theorem E are valid with $\|\cdot\|_{r,\mu}$ in place of $\|\cdot\|_r$, $T_{r,\mu}$ in place of T_r and the constants c_1, c_2, c_3 can be chosen independent of $\mu \in (0,1]$.

Proof. We merely comment on the changes in the proof of Theorem E necessary to prove Proposition 1.5. Proposition 1.3 remains correct if (1.27) is replaced by

$$(1.27)'\qquad \frac{1}{R^{2\mu}} \|u\|^{m-1}_{L^\infty(B_R)} \leq K\Big(A^{\lambda(m-1)}|u|^{\frac{2\lambda(m-1)}{N}}_{r,\mu} + |u|^{m-1}_{r,\mu}\Big)$$

and $|\cdot|_{r,\mu}$ is defined in the obvious way (see (1.25)). Indeed, put

$$(1.43)'\qquad A_\mu = \sup_{R\geq r} \frac{\|u\|^{m-1}_{L^\infty(B_R)}}{R^{2\mu}}$$

and use

$$\frac{1}{R^2}\int (\phi(\tfrac{x}{R})u)^p\, u^{m-1} \le 2^{2\mu}\, \frac{\|u\|^{m-1}_{L^\infty(B_{2R})}}{(2R)^{2\mu}}\, \frac{1}{R^{2(1-\mu)}} \int (\phi(\tfrac{x}{R})u)^p$$

$$\le c\, A_\mu \int (\phi(\tfrac{x}{R})u)^p$$

for $R > r > 1$ to conclude

$$(\int (\phi(\tfrac{x}{R})u)^{sp+b})^{1/4} \le c\, p\, (A + A_\mu) \int (\phi(\tfrac{x}{R})u)^p \quad . \tag{1.46)'$$

Then put $\theta_0 = \mu(\frac{2}{m-1} + N)$ in (1.49), so $\theta_k + \mu\, \frac{2}{m-1}$. The result will be (1.55) with A, $|u|_r$ replaced by A_μ, $|u|_{r,\mu}$.

Next, we go back to (1.28) which is valid with $g(t) = |u(t)|_{r,\mu}$ and c independent of $\mu \in (0,1]$. Combining with (1.6), we first obtain that

$$g(t) \le g(0)\, e^{c(|u_0|_r)t^{2\lambda/N}} \tag{1.68}$$

which proves that $|u(t)|_{r,\mu}$ remains uniformly bounded on $[0,T_r(u_0))$. Then, replace A by A_μ, $|\cdot|_r$ by $|\cdot|_{r,\mu}$ in all the computations (1.28) - (1.36) and use (1.27)' instead of (1.27) to obtain (1.67). The modified (1.8), (1.10) are obtained in a similar way.

In general, $T_{r,\mu}(u_0) < T_r(u_0)$ for $\mu \in (0,1)$ so that (1.67) does not directly prove that $\frac{u(t)}{(1+|x|^2)^{\mu/(m-1)}}$ remains bounded for $t \in (0,T_r(u_0))$. But for this one can use (1.68) and (1.27)'.

To complete the proof, observe that

$$X^\mu \subset X_0 \quad \text{for} \quad \mu \in (0,1)$$

whence $\lim_{r\uparrow\infty} T_r(u_0) = +\infty$.

<u>Remark 8</u>. In Proposition 1.5 we have excluded the case $\mu = 0$. Note that the definition (1.63) with $\mu = 0$ would lead to

$$\|f\|_{r,0} = \|f\|_{L^1(\mathbb{R}^N)} \quad \forall r > 0$$

and the conclusions of the proposition are in fact true with $\mu = 0$ (it is well-known that $u_0 \in L^1(\mathbb{R}^N) \Longrightarrow u(t) \in L^\infty(\mathbb{R}^N)$, see [5], [13]). But this is not the natural limiting case, which is in fact the result stated next. It provides a sufficient condition on u_0 (much more general than $u_0 \in L^1(\mathbb{R}^N)$) guaranteeing $U(t,u_0) \in L^\infty(\mathbb{R}^N)$ for all $t > 0$. Moreover this condition on u_0 is necessary by the results of [2].

<u>Proposition 1.3</u>. Let $u_0 \in L^1_{\ell oc}(\mathbb{R}^N)$ satisfy

$$\sup_{z\in\mathbb{R}^N} \int_{B(z,1)} |u_0| = \|u_0\|_0 < +\infty \quad , \tag{1.69}$$

where $B(z,1) = \{x \in \mathbb{R}^N; |x-z| < 1\}$. Then

$$\|U(t,u_0)\|_{L^\infty(\mathbb{R}^N)} \le \frac{c}{t^\lambda} \|u_0\|_0^{2\lambda/N} + c'\|u_0\|_0 \quad \text{for} \quad t > 0 \quad , \tag{1.70}$$

where c, c' depend only on N and m.

<u>Proof</u>. Again the proof is obtained by slightly modifying the proof of Theorem E. First, Proposition (1.1) remains true if (1.27) is replaced by

$$\|u\|^{m-1}_{L^\infty(\mathbb{R}^N)} \le K(\Lambda^{\lambda(m-1)}\|u\|_0^{\frac{2\lambda(m-1)}{N}} + \|u\|_0^{m-1}) \quad . \tag{1.27)"$$

Indeed put $\psi(x) = \phi_z(x) = \phi(x-z)$ in (1.39), (1.40), (1.41) to replace (1.42) by

$$\int \left|\nabla\phi_z(x)^{\frac{p+m-1}{2}}\right|^2 \le c\, p[\Lambda \int (\phi_z(x)u)^p + \int (\phi_z(x)u)^p u^{m-1}] \tag{1.42)"$$

and (1.44) by

$$\int \left|\nabla\phi_z(x)^{\frac{p+m-1}{2}}\right|^2 \le cp(\Lambda + A_0) \int (\phi_z(x)u)^p \quad , \tag{1.44)"$$

where

$$A_0 = \|u\|^{m-1}_{L^\infty(\mathbb{R}^N)} \quad .$$

For $N \ge 3$ we use Sobolev's inequality to obtain

$$\int (\phi_z(x)u)^{sp+b} \le c^s p^s (\Lambda + A_0)^s [\int (\phi_z(x)u)^p]^s \tag{1.48)"$$

with s, b defined as in (1.47). Then set

$$\begin{cases} p_0 = 1 \qquad p_{k+1} = p_k s + b \\ \\ a_k = \sup_{z\in\mathbb{R}^N} \int (\phi_z(x)u)^{p_k} \end{cases}$$

and finish as in the proof of Theorem E to obtain (1.27)" by noticing that

$$\|u\|^{m-1}_{L^\infty(\mathbb{R}^N)} = \sup_{z\in\mathbb{R}^N} \|u\|^{m-1}_{L^\infty(B(z,2))} \le c \limsup_{k\to\infty} (a_k)^{\frac{1}{p_k}} .$$

Next we go back to (1.28) and modify the arguments to obtain

$$g(t) \le g(0) + \int_0^t \|u\|^{m-1}_{L^\infty(\mathbb{R}^N)} g(\tau)d\tau$$

with $g(t) = \|u(t)\|_0$. Obviously adaptations of the rest of the proof of Theorem E lead to (1.70) with $c' = 0$ but for $t \in (0,c_1/\|u_0\|_0^{m-1})$. To complete the proof, one uses that $t \to \|U(t,u_0)\|_{L^\infty(\mathbb{R}^N)}$ is nonincreasing, so that

$$\forall\ t > 0,\ \|U(t,u_0)\|_{L^\infty(\mathbb{R}^N)} \le \left(\frac{c}{t^\lambda} + \frac{c}{t_0^\lambda}\right) \|u_0\|_0^{\frac{2\lambda}{N}},\ t_0 = c_1/\|u_0\|_0^{m-1}$$

which yields (1.70) with $c' = c/c_1^\lambda$.

Our last existence result concerns (IVP) when the initial datum is a Radon measure μ on $\mathbb{R}^N$ satisfying

(1.71) $$\sup_{R\ge r} \frac{1}{R^{\frac{2}{m-1}+N}} |\mu|\ (B_R) = \|\mu\|_r < \infty \quad \text{for} \quad r \ge 1 \ .$$

where $|\mu|$ is the variation of μ. We set

$$\ell(\mu) = \lim_{r\uparrow\infty} \|\mu\|_r \ .$$

<u>Proposition 1.6</u>. Let μ be a Radon measure on $\mathbb{R}^N$ satisfying (1.71) and c_1, c_2, c_3 the constants of Theorem 1.1. Then there is a function $u(x,t)$ defined for $0 < t < T(\mu) = c_1/\ell(\mu)^{m-1}$ such that

(a) $$\frac{\|u(\cdot,t)\|_{L^\infty(B_R)}}{R^{2/(m-1)}} \le \frac{c_2}{t^\lambda} \|\mu\|_r^{2\lambda/N}$$

for $R \ge r \ge 1$, $\quad 0 < t < T_r(\mu) = c_1/\|\mu\|_r^{m-1}$

(b) $\{t \mapsto u(\cdot,t)\} \in C((0,T(\mu));\ L^1_{\ell oc}(\mathbb{R}^N))$

(c) $\|u(\cdot,t)\|_r \le c_3\|\mu\|_r$ for $0 \le t < T_r(\mu)$

(d) For $\psi \in C_0^\infty(\mathbb{R}^N \times [0,T(\mu)))$

$$\int_0^{T(\mu)} \int (u\psi_t + |u|^{m-1}u\,\Delta\psi) = \int \psi(x,0)d\mu(x) \quad .$$

Proof. First observe tht (a), (b), (c) imply that $u(x,t)$ is a measurable, locally bounded function on $\mathbb{R}^N \times (0,T(\mu))$ such that, for $\theta \in C_0^\infty(\mathbb{R}^N)$ with $\theta(x) = 0$ on $\{|x| > R > r\}$ and $\theta > 0$

$$(1.72)\qquad \begin{aligned} \int_0^t \int |u|^m\theta &< \|\theta\|_\infty c_2^{m-1} R^2 \|\mu\|_r^{\frac{2\lambda(m-1)}{N}} \int_0^t \int_{B_R} |u(x,\tau)| \frac{d\tau}{\tau^{\lambda(m-1)}} \\ &< \|\theta\|_\infty\, c(R)\, \|\mu\|_r^\delta\, t^{-2\lambda/N} \quad , \quad \delta = \frac{2\lambda(m-1)}{N} + 1 \quad . \end{aligned}$$

Hence (d) makes sense.

Now let $\mu_n = \rho_n * \mu$ where ρ_n is a sequence of mollifiers on $\mathbb{R}^N$. Then $\mu_n \in X$ and $\|\mu_n\|_r$ converges to $\|\mu\|_r$ for all $r > 0$. Theorem E provides existence of $u_n(x,t) = U(t,\mu_n)$ satisfying (a), (b), (c), (d) with μ_n, u_n in place of μ and u. In particular, u_n is locally bounded, uniformly in n on $\mathbb{R}^N \times (0,T(\mu_n))$. Then, if one can prove that a subsequence of u_n converges a.e. (x,t), the limit u will satisfy the conclusions of Proposition 1.4 ((d) is obtained with the help of (1.72) which controls the behavior near $t = 0$). This a.e. convergence is a consequence of the compactness of u_n^m in $L^2_{loc}((0,T(\mu_n)) \times \mathbb{R}^N)$ as shown by the following remarks.

Let $R > 1$ and $\psi(x) = \phi(\frac{x}{R})$ where ϕ is chosen as in (1.24). Then multiplying (formally) (IVP) by $\psi\,|u|^{m-1}u$ (written $\psi\,u^m$ for simplicity) gives

$$\begin{aligned} \int \psi u^m u_t &= \int \psi u^m \Delta u^m = -\int u^m \nabla\psi\nabla u^m - \int \psi\,|\nabla u^m|^2 \\ &= \frac{1}{2}\int u^{2m}\Delta\psi - \int \psi|\nabla u^m|^2 \quad . \end{aligned}$$

We deduce for $0 < s < t < T(\mu_n)$

$$(1.73)\qquad \int_s^t \int_{\{|x|<R\}} |\nabla u^m|^2 < \frac{1}{2}\int_s^t \int_{\{|x|<2R\}} u^{2m} + \frac{1}{m+1}\int_{\{|x|<2R\}} u^{m+1}(s)$$

which proves that ∇u_n^m is uniformly bounded in $L^2_{\ell oc}(\mathbb{R}^N \times (0,T(\mu_n)))$. Then we multiply (IVP) by $\psi(u^m)_t$ and obtain

$$\int \psi(u^m)_t u_t = -\int (u^m)_t \nabla u^m \nabla\psi - \int \psi \nabla u^m \nabla(u^m)_t .$$

This implies

(1.74) $\int \psi(u^m)_t u_t + \frac{\partial}{\partial t}\frac{1}{2}\int \psi|\nabla u^m|^2 \le \|\frac{\nabla\psi}{\psi^{1/2}}\|_{L^\infty(\mathbb{R}^N)} (\int_{\{|x|<2R\}}|\nabla u^m|^2)^{1/2}(\int \psi[(u^m)_t]^2)^{1/2}$

where

(1.75) $$\|\frac{\nabla\psi}{\psi^{1/2}}\|_{L^\infty(\mathbb{R}^N)} \le c(R) < \infty$$

by (1.38) applied with $p = 2$. Now we remark that

(1.76) $$\int \psi[(u^m)_t]^2 \le m\|u^{m-1}(t)\|_{L^\infty(B_{2R})} \int \psi(u^m)_t u_t .$$

We use (1.74), (1.76), (1.73) to obtain

(1.77) $$a(t)^2 + \sigma(t)\frac{\partial}{\partial t}\int \psi|\nabla u^m|^2 \le c(R)\sigma(t)a(t)\left(\int_{B_{2R}}|\nabla u^m(t)|^2\right)^{1/2} ,$$

where we set $a(t) = [\int \psi[(u^m)_t)^2]^{1/2}$, $\sigma(t) = \frac{m}{2}\|u^{m-1}(t)\|_{L^\infty(B_{2R})}$. But

$$x^2 + y \le \alpha x \Longrightarrow y \le \frac{\alpha^2}{4} .$$

Thus, (1.77) implies

$$\frac{\partial}{\partial t}\int \psi|\nabla u^m|^2 \le \frac{c^2(R)}{4}\sigma(t)\int_{B_{2R}}|\nabla u^m(t)|^2 .$$

Coupled with (1.73), this proves that ∇u_n^m is uniformly bounded in $L^\infty_{\ell oc}((0,T(\mu_n));$ $L^2(B_R))$ for all $R > 0$. We use this in (1.77) to obtain

$$\int_s^t a^2(\sigma)d\sigma \le c(R,t,s) < +\infty \quad \text{for} \quad 0 < s < t < T(\mu_n) .$$

Finally, we have proved that u_n^m, ∇u_n^m; $(u_n^m)_t$ are bounded in $L^2_{\ell oc}(\mathbb{R}^N \times (0,T(\mu_n)))$ uniformly in n. This yields the compactness needed for u_n and completes the proof of Proposition 1.4 - provided the above computations are justified. We leave this last task to the reader.

Section 2: Uniqueness

We shall use the notation of Section 1.

Theorem U. Let $T > 0$ and $u : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ satisfying

(i) $u \in C([0,T]; L^1_{\ell oc}(\mathbb{R}^N)) \cap L^\infty(0,T;X)$,

(ii) $\forall \varepsilon > 0$, $u\rho_{1/(m-1)} \in L^\infty(\mathbb{R}^N \times [\varepsilon,T])$,

(iii) $u_t = \Delta|u|^{m-1}u$ in $\mathcal{D}'(\mathbb{R}^N \times (0,T))$.

Then

$$u(t) = U(t,u(0)) \quad \text{for} \quad 0 \le t \le \min(T,T(u(0))) \quad .$$

We begin by proving a weaker uniqueness result, namely:

Proposition 2.1. Let $T > 0$ and u, v satisfy

(i) $u, v \in C([0,T]; L^1_{\ell oc}(\mathbb{R}^N))$,

(ii) $u\rho_{1/(m-1)}, v\rho_{1/(m-1)} \in L^\infty(\mathbb{R}^N \times [0,T])$,

(iii) $u_t - \Delta|u|^{m-1}u = v_t - \Delta|v|^{m-1}v$ in $\mathcal{D}'(\mathbb{R}^N \times (0,T))$,

(iv) $(u-v)(0) = 0$.

Then $u = v$.

Remark 1. Proposition 2.1 has been proved in the case $N = 1$ by Kalashnikov in [8] with the extra assumption that u, v are continuous. We use similar methods here. All the other uniqueness results that have been obtained for (IVP) deal with initial data which do not grow when $|x| \to \infty$ (see [8], [10], [11], [12], [14]). Theorem U recovers all of them but those dealing with initial data which are measures (see [10], [12]).

Proof of Proposition 2.1. It suffices to prove $u(T) = v(T)$. For simplicity we will write u^m in place of $|u|^{m-1}u$. As a consequence of (iii), (iv), for all $\psi \in C_0^\infty(\mathbb{R}^N \times [0,T])$ we have

$$\int_0^T \int (u-v)\psi_t + (u^m - v^m)\Delta\psi = \int (u-v)(T)\psi(T) \quad .$$

If we set

$$\begin{aligned} a(x,t) &= \frac{u^m(x,t)-v^m(x,t)}{u(x,t)-v(x,t)} \quad \text{if} \quad u(x,t) \neq v(x,t) \\ &= mu^{m-1}(x,t) \qquad \text{if} \quad u(x,t) = v(x,t) \end{aligned}$$

the above equality becomes

$$\int_0^T \int (u-v)(\psi_t + a\Delta\psi) = \int (u-v)(T)\psi(T) \quad . \tag{2.1}$$

Now let $R_0 > 0$ be fixed and

$$\begin{cases} a_n \in C^\infty(\mathbb{R}^N \times [0,T]), \ a_n > 0, \ n = 1,2,\ldots, \\ \theta \in C_0^\infty(\mathbb{R}^N), \ 0 \le \theta \le 1, \ \theta(x) = 0 \ \text{ for } \ |x| > R_0) \quad . \end{cases} \tag{2.2}$$

Let $R > R_0 + 1$ and ψ_n be a solution of

$$\begin{cases} \psi_{nt} + a_n\Delta\psi_n = 0 \ \text{ in } \ 0 < t < T, \ |x| < R \quad , \\ \psi_n|_{|x|=R} = 0 \quad , \\ \psi_n(x,T) = \theta(x) \quad . \end{cases} \tag{2.3}$$

Finally, let $0 < \varepsilon < \frac{1}{2}$ and

$$\begin{cases} \phi_\varepsilon \in C_0^\infty(\mathbb{R}^N), \ 0 \le \phi_\varepsilon \le 1 \quad , \\ \phi_\varepsilon = 1 \ \text{ on } \ \{|x| < R-2\varepsilon\}, \ \phi_\varepsilon = 0 \ \text{ on } \ \{|x| > R-\varepsilon\} \quad , \\ \|\nabla\phi_\varepsilon\|_{L^\infty} \le \frac{c}{\varepsilon} \quad , \quad \|\Delta\phi_\varepsilon\|_{L^\infty} \le \frac{c}{\varepsilon^2} \quad . \end{cases} \tag{2.4}$$

Now put $\psi = \phi_\varepsilon\psi_n$ in (2.1) to get (recall $a(u-v) = u^m - v^m$)

$$\begin{aligned} &\int_0^T \int (u-v)\phi_\varepsilon(a-a_n)\Delta\psi_n + \int_0^T \int (u^m-v^m)(2\nabla\phi_\varepsilon\nabla\psi_n + \psi_n\Delta\phi_\varepsilon) \\ &\qquad = \int (u-v)(T)\theta \quad . \end{aligned} \tag{2.5}$$

We denote the first and second integrals above by $I_{n\varepsilon}$, $J_{n\varepsilon}$ respectively. Next estimate $J_{n\varepsilon}$ using (2.4):

$$|J_{n\varepsilon}| \le c \int_0^T \int_{\{R-2\varepsilon<|x|<R\}} |u^m-v^m| \left(\frac{|\nabla\psi_n|}{\varepsilon} + \frac{|\psi_n|}{\varepsilon^2}\right) \quad .$$

Since $\psi_n = 0$ on $\{|x| = R\}$

$$\sup_{\substack{R-2\varepsilon<|x|<R \\ 0<t<T}} |\psi_n(x,t)| \le \varepsilon \sup_{\substack{R-2\varepsilon<|x|<R \\ 0<t<T}} |\nabla\psi_n(x,t)|$$

and

$$\lim_{\varepsilon\downarrow 0}\Big(\sup_{\substack{R-2\varepsilon<|x|<R\\ 0<t<T}}|\nabla\psi_n(x,t)|\Big) = \sup_{\substack{|x|=R\\ 0<t<T}}|\nabla\psi_n(x,t)| = \sup_{\substack{|x|=R\\ 0<t<T}}\Big|\frac{\partial\psi_n}{\partial\nu}(s,t)\Big| \quad ,$$

where $\frac{\partial\psi_n}{\partial\nu}$ denotes the normal derivative of ψ_n on the sphere $\{|x| = R\}$. Hence

$$J_n = \lim_{\varepsilon\downarrow 0}\sup J_{n\varepsilon} \le c\cdot R^{N-1}\cdot\Big(\sup_{\substack{|x|=R\\ 0<t<T}}\Big|\frac{\partial\psi_n}{\partial\nu}(x,t)\Big|\Big)\cdot\lim_{\varepsilon\downarrow 0}\sup_{\substack{R-2\varepsilon<|x|<R\\ 0<t<T}}|u^m-v^m|(x,t) \quad ,$$

and, using (ii),

$$J_n \le c\,R^{N-1+\frac{2m}{m-1}}\Big(\sup_{\substack{|x|=R\\ 0<t<T}}\Big|\frac{\partial\psi_n}{\partial\nu}(x,t)\Big|\Big) \quad . \tag{2.6}$$

For $I_{n\varepsilon}$, we have

$$I_{n\varepsilon} \le I_n = \Big(\int_0^T\int_{\{|x|<R\}}|u-v|^2\,\frac{(a-a_n)^2}{a_n}\Big)^{1/2}\Big(\int_0^T\int a_n|\Delta\psi_n|^2\Big)^{1/2} \quad . \tag{2.7}$$

Let us now estimate $\sup_{\substack{|x|=R\\ 0<t<T}}\Big|\frac{\partial\psi_n}{\partial\nu}(x,t)\Big|$. Assume we have chosen a_n such that

$$a_n(x,t) \le K(1 + |x|^2) \quad \text{for} \quad 0 < t < T,\ x \in \mathbb{R}^N \quad . \tag{2.8}$$

For reasons which will become apparent, let α, β be such that

$$\beta > \frac{N-1}{2} + \frac{m}{m-1} \tag{2.9}$$

and

$$\alpha > 4NK\beta(\beta+1) \quad . \tag{2.10}$$

Set

$$\psi(x,t) = \frac{e^{\alpha(T-t)}}{(1+|x|^2)^{\beta}} \quad .$$

Then

$$|a_n\Delta\psi| \le K(1+|x|^2)e^{\alpha(T-t)}\left|\frac{-2\beta N}{(1+|x|^2)^{\beta+1}} + \frac{4\beta(\beta+1)|x|^2}{(1+|x|^2)^{\beta+2}}\right|$$

$$\le 4NK\beta(\beta+1)\psi \quad .$$

Since $\psi_t = -\alpha\psi$, by (2.10) and above inequality

$$\psi_t + a_n\Delta\psi \le 0 \quad .$$

Thus, if we now choose λ so that

(2.11) $$\psi_n(x,T) = \theta(x) \le \frac{\lambda}{(1+|x|^2)^\beta} = \lambda\ \psi(T),\ |x| \le R \quad ,$$

by the maximum principle we will have

(2.12) $$\lambda\psi \ge \psi_n \quad \text{for} \quad 0 < t < T,\ |x| \le R \quad ,$$

which provides a first estimate for ψ_n. Note that (2.11) is satisfied if $\lambda = \|\theta\|_\infty(1+R_0^2)^\beta$.

Let us now construct a function g on the set $\{(x,t);\ R-1 \le |x| \le R,\ 0 < t < T\}$ such that

(2.13) $$g \ge \psi_n \quad \text{and} \quad g(x,t) = 0 \quad \text{for} \quad |x| = R,\ 0 < t < T \quad .$$

By (2.3), (2.13) we will then have

$$\frac{\partial}{\partial\nu}(g-\psi_n)(x,t) \le 0 \quad \text{for} \quad |x| = R,\ 0 < t < T$$

and hence (recall that $\frac{\partial\psi_n}{\partial\nu} \le 0$)

(2.14) $$\sup_{\substack{|x|=R\\0<t<T}} \left|\frac{\partial\psi_n}{\partial\nu}(x,t)\right| \le \sup_{\substack{|x|=R\\0<t<T}} \left|\frac{\partial g}{\partial\nu}(x,t)\right| \quad .$$

If $N \ge 3$, let g, which is independent of t, be defined by

$$g(x,t) = \frac{d}{|x|^{N-2}} + e = g(x)$$

where e, d satisfy

(2.15) $$\begin{cases} \dfrac{d}{(R-1)^{N-2}} + e = \dfrac{\lambda e^{\alpha T}}{(1+(R-1)^2)^\beta} \\[2ex] \dfrac{d}{R^{N-2}} + e = 0 \quad . \end{cases}$$

Note that $\Delta g = 0$ on $\{R-1<|x|<R\}$. Moreover, by (2.12), (2.15), (2.3)

$$g(x) \geq \psi_n(x,t) \quad \text{for} \quad |x| = R-1,\ 0 < t < T \ ,$$

$$g(x) \geq \psi_n(x,T) = 0 \quad \text{for} \quad R-1 < |x| < R \ ,$$

$$0 = g(x) = \psi_n(x,t) \quad \text{for} \quad |x| = R,\ 0 < t < T \ .$$

Therefore (2.13) holds by maximum principle and so does (2.14). It remains to estimate $\frac{\partial g}{\partial \nu}$,

$$\frac{\partial g}{\partial \nu}(x) = \frac{(2-N)d}{R^{N-1}} = \frac{2-N}{R^{N-1}} \frac{\lambda e^{\alpha T}}{(1+(R-1)^2)^{\beta}} \left(\frac{1}{(R-1)^{N-2}} - \frac{1}{R^{N-2}}\right)^{-1} \ .$$

Hence

$$\frac{\partial g}{\partial \nu}(x) \leq \frac{c}{R^{2\beta}} \quad \text{for} \quad |x| = R \ , \tag{2.16}$$

where c does not depend on $R \geq 2$. The same computations with $g(x) = d\ell n|x| + e$ if $N = 2$ and $g(x) = d|x| + e$ if $N = 1$ lead to the same estimate (2.16). Combining with (2.14), we have

$$\sup_{\substack{|x|=R \\ 0<t<T}} \left|\frac{\partial \psi_n}{\partial \nu}(x,t)\right| \leq \frac{c}{R^{2\beta}} \ . \tag{2.17}$$

Going back to (2.6), we obtain

$$J_n \leq cR^{N-1+\frac{2m}{m-1}-2\beta} \tag{2.18}$$

where c depends only on m, N, θ, R_0 and K defined in (2.8).

To estimate I_n, multiply the equation in (2.3) by $\Delta\psi_n$ and integrate to obtain

$$\frac{1}{2}\int |\nabla\psi_n(0)|^2 + \int_0^T \int a_n|\Delta\psi_n|^2 = \frac{1}{2}\int |\nabla\theta|^2 \ .$$

Thus (see 2.7)

$$I_n \leq c(R) \left[\int_0^T \int_{\{|x|<R\}} \frac{(a-a_n)^2}{a_n}\right]^{1/2} \ . \tag{2.19}$$

Let us choose $a_n = \bar{a}*\rho_n + \frac{1}{n}$ where $\bar{a}$ is the extension by 0 of a to $\mathbb{R} \times \mathbb{R}^N$ and ρ_n a sequence of mollifiers in $\mathbb{R} \times \mathbb{R}^N$ such that

$$\int_0^T \int_{\{|x|<R\}} (a-\bar{a}*\rho_n)^2 \leq \frac{1}{n^2} \ .$$

Then, a_n satisfies (2.2), $a_n > \frac{1}{n}$, and a_n satisfies (2.8) with

$$K = 1 + m \max\left(\|u^{m-1}\rho\|_{L^\infty(\mathbb{R}^N\times(0,T))}, \|v^{m-1}\rho\|_{L^\infty(\mathbb{R}^N\times(0,T))}\right)$$

which is finite by assumption. Moreover

$$[\int_0^T \int_{\{|x|<R\}} \frac{(a-a_n)^2}{a_n}]^{1/2} < \sqrt{n}\ [\frac{1}{n} + \frac{(TR^N)^{1/2}}{n}] = \frac{c(R)}{\sqrt{n}} \quad . \tag{2.20}$$

Now, for R fixed, $R > R_0 + 1$, we let n tend to ∞ in (2.6) and (2.7). Using (2.5), (2.18), (2.19), (2.20), we obtain

$$|\int (u-v)(T)\theta| < c\, R^\sigma \quad , \quad \sigma = N-1 + \frac{2m}{m-1} - 2\beta \quad .$$

Letting R tend to ∞ and using (2.9) finally yield

$$|\int (u-v)(T)\theta| < 0 \quad .$$

Since θ is arbitrary we deduce $u = v$.

<u>Proof of Theorem U</u>

Let u satisfy the assumptions of Theorem U. By Theorem E and Proposition 2.1, for all $\varepsilon > 0$ small enough

$$u(t+\varepsilon) = U(t,u(\varepsilon)) \tag{2.21}$$

for all $0 < t < \min(T-\varepsilon, T(u(\varepsilon)))$. Thanks to the continuity property stated in Theorem E, (1.9), Theorem U will be proved by letting ε go to 0 in (2.21), provided one can show that $u(\varepsilon)$ *converges to* $u(0)$ *in* $L^1(\rho_\alpha)$ for some $\alpha > 0$ (note that $T_r(u(0)) > \limsup_{\varepsilon\downarrow 0} T_r(u(\varepsilon))$). But, by (2.21) and Theorem E

$$\left|\frac{u^{m-1}(x,t+\varepsilon)}{1+|x|^2}\right| < \frac{c_\varepsilon}{t^{\lambda(m-1)}},\ x \in \mathbb{R}^N,\ 0 < t < \min(T-\varepsilon, T_1(u(\varepsilon))) \quad , \tag{2.22}$$

where c_ε depends on $\|u(\varepsilon)\|_X$. Since $u \in L^\infty(0,T:X)$, one can pass to the limit in (2.22) and obtain

$$\left|\frac{u^{m-1}(x,t)}{1+|x|^2}\right| < \frac{c}{t^{\lambda(m-1)}} \tag{2.23}$$

where c does not depend on ε. On the other hand, as noticed in the proof of the estimates (1.8), (1.9), for all $\psi \in C_0^\infty(\mathbb{R}^N)$, $\psi \geq 0$, one has

$$\frac{d}{dt}\int \psi|U(t,u(\varepsilon))| \leq \int \Delta\psi|U(t,u(\varepsilon))|^m \quad \text{on} \quad (0,T(u(\varepsilon)))$$

for all $\varepsilon > 0$ small. By (2.21) this gives

$$\frac{d}{dt}\int \psi|u(t)| \leq \int \Delta\psi|u|^m \quad \text{for} \quad t \in (0,\alpha),\ \alpha > 0 \ \text{and small} \quad .$$

Thanks to (2.23), we may integrate this up to 0 and get

$$\int \psi|u(t)| \leq \int \psi|u(0)| + c\int_0^t \frac{d\tau}{\tau^{\lambda(m-1)}}\int \Delta\psi(1+|x|^2)|u(\tau)| \quad . \tag{2.24}$$

Now we choose $\alpha > 1 + \frac{1}{m-1} + \frac{N}{2}$. By Lemma 1.1

$$\|(1+|x|^2)u(\tau)\|_{L^\infty(0,T;\ L^1(\rho_\alpha))} \leq c\|u\|_{L^\infty(0,T;X)} < +\infty \quad . \tag{2.25}$$

We put $\psi = \rho_\alpha\theta(\frac{|x|}{n})$ in (2.24) where $\theta \in C_0^\infty(\mathbb{R})$, $\theta(r) = 1$ for $r \in [0,1]$ and we let n tend to ∞. Thanks to (2.25) we obtain, in the limit,

$$\int \rho_\alpha|u(t)| \leq \int \rho_\alpha|u(0)| + cc_\alpha\int_0^t \frac{d\tau}{\tau^{\lambda(m-1)}}\int \rho_\alpha|u(\tau)| \quad , \tag{2.26}$$

where we used $|\Delta\rho_\alpha|(1+|x|^2) \leq c_\alpha\rho_\alpha$. From (2.26), (2.25) we deduce

$$\limsup_{\varepsilon\downarrow 0}\int \rho_\alpha|u(\varepsilon)| \leq \int \rho_\alpha|u(0)| \quad .$$

Combine this with Fatou's lemma and one obtains that $\int \rho_\alpha|u(\varepsilon)|$ converges to $\int \rho_\alpha|u(0)|$. Since $u(\varepsilon)$ converges to $u(0)$ in $L^1_{loc}(\mathbb{R}^N)$, by Lebesgue's theorem, $u(\varepsilon)$ converges to $u(0)$ in $L^1(\rho_\alpha)$. This completes the proof of Theorem U.

Combining Theorem E and Theorem U, we can state

<u>Theorem EU</u>. Let $u_0 \in X$. Then, there exists $T \in (0,\infty]$ and a unique maximally defined solution u of

(i) $u \in C([0,T);\ L^1_{loc}(\mathbb{R}^N)) \cap L^\infty_{loc}([0,T);\ X)$,

(ii) $u\rho_{1/(m-1)} \in L^\infty_{loc}(\mathbb{R}^N \times (0,T))$,

(iii) $u_t = \Delta|u|^{m-1}u$ in $\mathcal{D}'(\mathbb{R}^N \times (0,T))$,

(iv) $u(0) = u_0$.

Moreover, if $T < \infty$,

(a) $\lim_{t\uparrow T}\|u(t)\|_r = \infty$, $\forall r > 0$

(b) $\lim_{t \uparrow T} \|u^{m-1}(t)\rho\|_{L^\infty(\mathbb{R}^N)} = \infty$,

(c) $\lim_{t \uparrow T} \int_0^t \|u^{m-1}(t)\rho\|_{L^\infty(\mathbb{R}^N)} = \infty$.

<u>Proof</u>. Clearly, there exists a maximally defined u on some $[0,T)$ with properties (i), (ii), (iii), (iv). The uniqueness is a direct consequence of Theorem U.

Assertion (a) must hold, for if there is a sequence of times $t_n \uparrow T$ such that $\|u(t_n)\|_r \leq M < \infty$, we can define

$$v(t) = \begin{cases} u(t) , & t < t_n \\ U(t-t_n, u(t_n)) , & t_n < t < t_n + c_1/M^{m-1} \end{cases} .$$

But v extends u past T for large n.

Assertion (b) is a direct consequence of (a). The last assertion follows from (1.26) and (a).

Appendix X

We begin with the proof of Lemma 1.2.

Proof of Lemma 1.2. For (i), let $f \in X$ and

$$\frac{1}{R^{\frac{2}{m-1}+N}} \int_{\{|x|<R\}} |f| \le c \quad \text{for} \quad R \ge 1 \ . \tag{A.1}$$

Now we have

$$\int \frac{|f(x)|}{(1+|x|^2)^{\alpha}} \le \int_{\{|x|\le 1\}} |f(x)| + \int_{\{|x|>1\}} \frac{|f(x)|}{|x|^{2\alpha}} \tag{A.2}$$

and

$$\frac{1}{|x|^{2\alpha}} = \frac{1}{2\alpha} \int_{|x|}^{\infty} \frac{dR}{R^{2\alpha+1}} \ .$$

Thus Fubini's theorem and (A.1) yield

$$\int_{\{|x|>1\}} \frac{|f(x)|}{|x|^{2\alpha}}\, dx = \frac{1}{2\alpha} \int_1^{\infty} \frac{1}{R^{2\alpha+1}} \left(\int_{1<|x|<R} |f(x)|\, dx\right) dR$$

$$\le \frac{c}{2\alpha} \int_1^{\infty} \frac{R^{\frac{2}{m-1}+N}}{R^{2\alpha+1}}\, dR = \frac{c}{4\alpha(\alpha - \frac{1}{m-1} - \frac{N}{2})} \ .$$

We then use (A.2) to obtain $X \subset L^1(\rho_{\alpha})$ with continuous embedding.

To prove (ii), observe that $f_n \to f$ in $L^1_{loc}(\mathbb{R}^N)$ implies that for $R \ge r$

$$\frac{1}{R^{\frac{2}{m-1}+N}} \int_{\{|x|<R\}} |f| \le \liminf_{n\to\infty} \frac{1}{R^{\frac{2}{m-1}+N}} \int_{\{|x|<R\}} |f_n|$$

$$\le \liminf_{n\to\infty} \|f_n\|_r$$

and the result follows upon taking the supremum over $R \ge r$ in this inequality.

The next result is used in Proposition A.2 to establish the equivalence of (0.3) and (0.3)' of the introduction.

Proposition A.1. Let $\lambda \in \mathbb{R}$, $\theta, \delta > 0$. For $u \in L^1_{loc}(\mathbb{R}^N)$, $u > 0$, define

$$I(u) = \sup_{R>1} \frac{1}{R^\theta} \int_{\{|x|<R\}} \frac{u(x)}{(1+|x|^2)^{\lambda+\delta}}$$

$$J(u) = \sup_{R>1} \frac{1}{R^{\theta+2\delta}} \int_{\{|x|<R\}} \frac{u(x)}{(1+|x|^2)^{\lambda}} .$$

Then, there exists $c_1 = c_1(\delta) < \infty$ such that

$$J(u) \le c_1 I(u) . \tag{A.3}$$

If $\theta > 0$, there exists $c_2 = c_2(\delta,\theta) < \infty$ such that

$$I(u) \le c_2 J(u) . \tag{A.4}$$

Proof.

The estimate (A.3) follows from the fact that for $R > 1$, $|x| < R$,

$$\frac{1}{R^{2\delta}} \le \left(\frac{1+R^2}{R^2}\right)^\delta \frac{1}{(1+R^2)^\delta} \le 2^\delta \frac{1}{(1+|x|^2)^\delta} .$$

It is sufficient to prove (A.4) for $\lambda = 0$ (then apply it to $\frac{u}{(1+|x|^2)^\lambda}$ instead of u). One can also assume $u(x) = 0$ for $|x| \le 1$. Indeed if $u = u_1 + u_2$ with

$$u_1(x) = \begin{cases} 0 & \text{if } |x| \le 1 \\ u(x) & \text{if } |x| > 1 \end{cases} \qquad u_2(x) = \begin{cases} u(x) & \text{if } |x| \le 1 \\ 0 & \text{if } |x| > 1 \end{cases}$$

then

$$I(u) \le I(u_1) + I(u_2)$$

and

$$I(u_2) = \int_{\{|x|\le 1\}} \frac{u(x)}{(1+|x|^2)^\delta} \le \int_{\{|x|\le 1\}} u(x) = J(u_2) \le J(u) .$$

Thus we assume $\lambda = 0$ and $u(x) = 0$ on $\{|x| \le 1\}$. We use that for all $R > 0$

$$\int_{\{\frac{R}{2} \le |x| \le R\}} \frac{u(x)}{(1+|x|^2)^\delta} \le \frac{1}{\left(1+\frac{R^2}{4}\right)^\delta} \int_{\{|x|\le R\}} u(x) \le 4^\delta R^\theta \frac{R^{2\delta}}{(4+R^2)^\delta} J(u) .$$

Hence for all $n \geq 0$

$$\int_{\{\frac{R}{2^{n+1}} \leq |x| \leq \frac{R}{2^n}\}} \frac{u(x)}{(1+|x|^2)^\delta} \leq 4^\delta (\frac{R}{2^n})^\theta \, J(u) \quad .$$

Summing these inequalities over all $n \geq 0$ gives

$$\int_{\{|x| \leq R\}} \frac{u(x)}{(1+|x|^2)^\delta} \leq 4^\delta R^\theta \, \frac{1}{1-2^{-\theta}} \, J(u) \quad ,$$

whence (A.4).

Proposition A.2

(i) $X = \{f \in L^1_{loc}(\mathbb{R}^N);\ \sup\limits_{R \geq 1} \frac{1}{R^N} \int_{\{|x| \leq R\}} |f(x)| \rho_{1/(m-1)}(x) < \infty\}$.

(ii) $\{f \in L^1_{loc}(\mathbb{R}^N);\ f\rho_{1/(m-1)} \in L^\infty(\mathbb{R}^N)\} \subset X$.

(iii) $L^1(\rho_{\frac{1}{m-1} + \frac{N}{2}}) \subset X$.

(iv) More generally, for $1 \leq p < \infty$

$$\{f \in L^p_{loc}(\mathbb{R}^N);\ f\rho_{1/(m-1)} \in L^p(\rho_{N/2})\} \subset X \quad .$$

Proof. For (i), apply Proposition (A.1) with $\theta = N$, $\delta = \frac{1}{m-1}$, $\lambda = 0$ and recall (1.2), (1.3). Then (ii) follows from (i). To obtain (iii), apply (A.4) with $\theta = \lambda = 0$ and $\delta = \frac{1}{m-1} + \frac{N}{2}$. For (iv), set

$$(|f|\rho_{1/(m-1)})^p \, \rho_{N/2} = w \in L^1(\mathbb{R}^N) \quad .$$

Then

$$\int_{\{|x| \leq R\}} |f| \leq R^{N(1-\frac{1}{p})} \left(\int_{\{|x| \leq R\}} |f|^p\right)^{1/p} \leq R^{N(1-\frac{1}{p})} (1+R^2)^{\frac{1}{m-1} + \frac{N}{2p}} \left(\int w\right)^{1/p}$$

$$\leq c \, R^{N + \frac{2}{m-1}} \quad .$$

We now identify X_0.

Proposition A.3. The space $X_0 = \{u \in X;\ \lim\limits_{r \uparrow \infty} \|u\|_r = 0\}$ is the closure of $L^1(\mathbb{R}^N)$ in X.

Proof. Let $f \in L^1(\mathbb{R}^N)$ and set $\gamma = \frac{2}{m-1} + N$. Since

$$\frac{1}{R^\gamma}\int_{\{|x|<R\}}|u| \le \frac{1}{R^\gamma}\int_{\{|x|<R\}}|u-f| + \frac{1}{R^\gamma}\int_{\{|x|<R\}}|f| \quad .$$

we have

$$\|u\|_r \le \|u-f\|_r + \frac{1}{r^\gamma}\|f\|_{L^1(\mathbb{R}^N)} \quad .$$

Since $\|\cdot\|_r$ decreases with r, for $r_0 > 0$

$$\limsup_{r\to\infty} \|u\|_r \le \|u-f\|_{r_0}$$

and

$$\limsup_{r\to\infty} \|u\|_r \le \inf\{\|u-f\|_{r_0} : f \in L^1(\mathbb{R}^N)\} \quad .$$

Thus X_0 contains the closure of $L^1(\mathbb{R}^N)$.

Conversely, assume $\lim_{r\to\infty} \|u\|_r = 0$. Then, if 1_r is the characteristic function of $\{|x|<r\}$, $u.1_r$ converges to u in X when r tends to ∞ (whence the result since $u.1_r \in L^1(\mathbb{R}^N)$). Indeed

$$\frac{1}{R^\gamma}\int_{\{|x|<R\}}|u-u.1_r| = \frac{1}{R^\gamma}\int_{\{r<|x|<R\}}|u| \le \|u\|_r \quad \text{for all} \quad R > 1 \quad .$$

<u>Proposition A.4</u>.

(i) $\{f \in L^1_{\ell oc}(\mathbb{R}^N);\ \lim_{|x|\to\infty} |f(x)|\rho_{1/(m-1)}(x) = 0\} \subset X_0 \quad .$

(ii) Let $1 < p < \infty$ and $\theta \in L^\infty_{\ell oc}(\mathbb{R}^N)$ satisfy $\theta > 0$ and $\lim_{|x|\to\infty} \theta(x) = \infty$. Then

$$\{f \in L^p_{\ell oc}(\mathbb{R}^N);\ f\rho_{1/(m-1)} \in L^p(\theta\rho_{N/2})\} \subset X_0 \quad ,$$

(iii) For $1 < p < \infty$, $L^p(\rho_{\frac{1}{m-1}+\frac{N}{2}}) \subset X_0 \quad .$

<u>Proof</u>. For (ii) let $w = [|f|\rho_{1/(m-1)}]^p\, \theta\rho_{N/2}$. Then

$$\int_{\{|x|<R\}} |f| \le \int_{\{|x|<r_0\}} |f| + R^{N(1-\frac{1}{p})} [\int_{\{r_0<|x|<R\}} |f|^p]^{1/p}$$

$$\le \int_{\{|x|<r_0\}} |f| + R^{N(1-\frac{1}{p})} (1+R^2)^{\frac{1}{m-1}+\frac{N}{2p}} (\sup_{r_0<|x|} (\theta(x))^{-1})(\int w)^{1/p} .$$

Hence

$$\|f\|_r \le \frac{1}{r^{\frac{2}{m-1}+N}} \int_{\{|x|<r_0\}} |f| + c \sup_{r_0<|x|} (\theta(x))^{-1} ,$$

and we deduce (ii). The assertion (i) is obtained similarly. For (iii) note that

$$|f|^p \rho_{\frac{1}{m-1}+\frac{N}{2}} = [|f| \rho_{\frac{1}{m-1}}]^p \rho_{\frac{1-p}{m-1}} \rho_{\frac{N}{2}}$$

and use (iii) with $\theta(x) = (1+|x|^2)^{\frac{p-1}{m-1}}$.

<u>Proposition A.5</u>.

$$\{f \in X;\ f\rho_{1/(m-1)} \in L^\infty(\mathbb{R}^N)\} \quad \text{is not dense in} \quad X \ .$$

<u>Proof</u>. By Proposition A.2 *(i) and Proposition* A.1*, the claim is equivalent to saying that* $L^\infty(\mathbb{R}^N)$ is not dense in

$$\{v \in L^1_{\ell oc}(\mathbb{R}^N);\ \sup_{R>1} \frac{1}{R^N} \int_{\{|x|<R\}} |v| < \infty\}$$

with the obvious norm. Let

$$v(x) = \begin{cases} n & \text{on} \quad n - \frac{1}{n} \le |x| \le n \qquad n = 1,2,\ldots \\ 0 & \text{otherwise} \ . \end{cases}$$

If $M-1 < R \le M$

$$\frac{1}{R^N} \int_{\{|x|<R\}} |v(x)| \le \frac{c}{(M-1)^N} \sum_1^M n^{N-1} \le c' < +\infty \ .$$

while if $f \in L^\infty(\mathbb{R}^N)$, $|f| \le k$,

$$\frac{1}{m^N}\int_{\{|x|<m\}}|v-f| \geq \frac{c}{m^N}\sum_{n=k}^{n=m} n^{N-2}(n-k)$$

$$\geq \frac{c}{m^N}(m^N - k^N - km^{N-1}) \quad .$$

Hence

$$\sup_{m\geq 1}\frac{1}{m^N}\int_{\{|x|<m\}}|v-f| \geq c > 0,\ \forall f \in L^\infty(\mathbb{R}^N) \quad .$$

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It is shown that the initial-value problem $u_t = \Delta(|u|^{m-1}u)$, $u(x,0) = u_0(x)$, where $m > 1$, has a solution on $\mathbb{R}^N \times (0,T)$ for some $T > 0$ if $R^{-(\frac{2}{m-1}+N)} \int_{\{|x|\le R\}} |u_0(x)|dx$ is bounded independently of $R \geq 1$. The restriction on u_0 is known to be necessary as well by recent results of Aronson and Cafarelli, so this theorem is the best possible. Many supplementary

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ABSTRACT (continued)

results give refined estimates on the solution under various conditions on u_0, establish uniqueness within the existence class, allow u_0 to be a Radon measure, establish continuous dependence on u_0 in various spaces, etc.

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